

Colored Generalization of Bulgarian Solitaire

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Abstract

This article introduces a colored generalization of Bulgarian solitaire, extending the classical combinatorial puzzle to the setting where partitions have distinguishable parts ($\mathcal{P}_n^\star$). We define four natural actions (T_1 , T_2 , T_3 , and T_4) on colored partitions, corresponding to different methods of cell removal and replacement in the Ferrers diagram representation.

We establish fundamental properties of these actions, including the fact that the reversal operation (rev) on partitions is an equivariant bijection between the actions T_1 and T_3 as well as between T_2 and T_4 . For actions T_1 and T_3 , we derive explicit formulas showing the orbit size equals the least common multiple of integers from 1 to $m + 1$.

For the more complex action T_2 , we develop a novel coloring algorithm that partitions the Ferrers diagram into distinct colored regions, providing a combinatorial method to determine the permutation σ_π induced by repeated application of T_2^{m+1} . We prove that the permutation σ_π^{col} defined by our coloring algorithm equals σ_π , and establish a precise formula for its order. Based on this analysis, we formulate a characterization of orbit sizes under T_2 that incorporates both the combinatorial structure of the colored partition and the dynamics of the underlying unlabeled Bulgarian solitaire.

1 Introduction

Bulgarian solitaire is a simple mathematical puzzle that has captivated mathematicians for decades. Its rules are easy to grasp where you repeatedly take one object from each pile to form a new pile. This process hides a surprising depth, connecting to advanced fields like partition theory and dynamical systems.

The story begins in May 1980, when the Russian mathematician Anatolii A. Karatsuba visited the Bulgarian Academy of Sciences and introduced the core idea. The puzzle entered the mathematical mainstream just a year later. In 1981, Andrei Toom provided the first rigorous proof, showing how the game inevitably settles into a “triangular” configuration for certain numbers of objects [19]. At the same time, B. Bojanov was discussing the problem within Bulgarian mathematical circles, helping to establish its foundations [3].

The puzzle’s intrigue grew quickly. In 1982, Jørgen Brandt delved into the number of distinct cycles that the card piles could form [4]. Then in 1983, Martin Gardner featured Bulgarian solitaire in his *Scientific American* column, catapulting it to international fame [7]. This exposure made it a popular topic in programming seminars, like one at Stanford led by Hobby and Knuth [11].

The mid-1980s saw a flurry of formal analysis. E. Akin and M. Davis published an English-language analysis for a broad mathematical audience [1], while proofs of the central “Bulgarian solitaire conjecture” were advanced by K. Igusa [16] and solidified by H.-J. Bentz [2].

Researchers continued to uncover its rich structure in the 1990s. From 1991 to 1998, G. Etienne revealed fascinating connections to Young tableaux [6], and Griggs and Ho further studied the cycling of partitions [8]. This combinatorial exploration continued into the 2000s, with Andrew V. Sills providing new proofs related to the puzzle [18].

A significant new direction emerged in 2004–2005 when Serguei Popov introduced a *random* version, linking the solitaire to Markov chains and probability [17]. This stochas-

tic variant opened up fresh avenues of research, which were explored by Brian Hopkins and others from 2006 to 2010, including studies of “Garden of Eden” configurations [14, 15].

A renewed wave of interest around 2012 led to modern re-examinations of Toom’s original proof [9] and a retrospective by Hopkins titled “30 years of Bulgarian Solitaire” [12]. Research expanded into stochastic versions and higher-dimensional generalizations, such as work by Eriksson and Sjöstrand on their limiting shapes [5]. Most recently, in 2024, Hopkins discussed a variant by Henrik Eriksson, demonstrating the puzzle’s ongoing vitality [13].

From 2018 to the present, Bulgarian solitaire remains a vibrant area of research, inspiring new discoveries across combinatorics, dynamical systems, and probability theory, and even finding its way into educational applications. In this project,

1. We reformulate Bulgarian solitaire in the language of partitions and dynamical systems, following the notation established in [18, 10], and introduce the concept of *partitions with distinguishable parts* ($\mathcal{P}_n^\star$).
2. We define four natural actions (T_1 , T_2 , T_3 , and T_4) on colored partitions, corresponding to different methods of cell removal and replacement in the Ferrers diagram representation.
3. We establish fundamental properties of these actions, including the fact that the reversal operation (rev) on partitions is an equivariant bijection between the actions T_1 and T_3 as well as between T_2 and T_4 . This implies that $|O_{T_1}(\pi)| = |O_{T_3}(\pi)|$ and $|O_{T_2}(\pi)| = |O_{T_4}(\pi)|$ for all $\pi \in \mathcal{P}_n^\star$.
4. We derive explicit formulas for orbit sizes under T_1 and T_3 , showing that the orbit size equals the least common multiple of the integers from 1 to $m + 1$.
5. For the more complex actions T_2 and T_4 , we develop a novel coloring algorithm that partitions the Ferrers diagram into distinct colored regions. This algorithm allows us to define a permutation σ_π^{col} on the labeled cells, and we prove that σ_π^{col} equals the permutation σ_π induced by T_2^{m+1} (Theorem 5).
6. We compute the order of σ_π (Theorem 6) and provide a combinatorial interpretation of the cycle structure through our coloring algorithm. Based on this analysis, we formulate a conjecture for the exact orbit size of T_2 (Conjecture 1), supported by examples and partial results.

The paper is organized as follows: Section 2 provides necessary background on classical Bulgarian solitaire and partition theory. Section 3 introduces our colored generalization and defines the four actions $T_1, \dots, T_4$. Section 4 establishes basic properties and relations between these actions. Sections 4.1 – 4.2 analyzes orbit sizes, with particular attention to the combinatorial structures emerging from T_2 . Sections 4.3 – 4.4 presents our coloring algorithm and proves its correctness.

2 Background: Bulgarian Solitaire

2.1 Partitions and Ferrers Diagrams

The dynamics of Bulgarian solitaire are naturally expressed in the language of integer partitions. The problem of Bulgarian solitaire is reformulated into the language of partitions

and dynamical systems. The usage in [18, 10] is followed here.

A partition π of a positive integer n is a decreasing sequence of nonnegative integers such that the sum of terms is n , that is,

$$\pi = (\pi_1, \pi_2, \pi_3, \dots)$$

where π_i is a nonnegative integer, $\pi_i \geq \pi_{i+1}$ for all i , and $\sum_{i=1}^{\infty} \pi_i = n$.

Since π_i is a nonnegative integer, only a finite number of terms of π are nonzero. These are called the parts of π , and the number of parts of π is called the length of π and is denoted by $l(\pi)$. In writing specific partitions one typically omits the infinite tail of zeros. For example, the partition $\pi = (3, 1, 1, 0, 0, \dots)$ is written as $(3, 1, 1)$. We can represent it using a rotated Ferrers diagram, as in Figure 1.

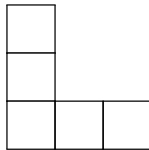


Figure 1: Ferrers diagram of $\pi = (3, 1, 1) \in \mathcal{P}_5$

We denote the set of all partitions by $\mathcal{P}$, and the subset of all partitions of n by $\mathcal{P}_n$. For two partitions μ and π in $\mathcal{P}$, we define the union $\mu \cup \pi$ to be the partition obtained by merging the entries of μ with those of π and arranging the resulting entries in decreasing order, for example,

$$(3, 1, 1) \cup (5, 4, 2, 1) = (5, 4, 3, 2, 1, 1, 1).$$

2.2 The Bulgarian Solitaire Operation

Bulgarian solitaire is a discrete dynamical system defined on $\mathcal{P}_n$. The system evolves through an iterative process governed by the following rule: from the current partition, remove one cell from each existing pile (column) to form a new pile. Formally, Bulgarian solitaire is the action defined by the map $T : \mathcal{P} \rightarrow \mathcal{P}$ via

$$T(\pi) = (\pi_1 - 1, \pi_2 - 1, \dots, \pi_{l(\pi)} - 1, 0, 0, \dots) \cup (l(\pi), 0, 0, \dots).$$

Since $\mathcal{P}_n$ is finite, iterating T from any initial partition π must eventually lead to a cycle. The resulting sequence $O_T(\pi) = \{\pi, T(\pi), T^2(\pi), \dots\}$ is the *orbit* of π .

Hart, Khan, and Khan [10] give the following visual description of the action, which helps them to give an exposition of Toom's proof of Bulgarian Solitaire that appeared in Kvant[19].

1. We remove all of the cells in the bottom row of the Ferrers diagram.
2. We translate the remaining cells down one square and then one square to the right.
3. We now place the cells that we had removed from the bottom row into the first column. In doing so, we take care to place the cell that originally was on the position $(k, 1)$ onto the position $(1, k)$.

4. Finally if $l(\pi) < \pi_1 - 1$, then we move the cells in the rows numbered $l(\pi) + 1$ through $\pi_1 - 1$, left one square.

We note that if $\pi \in \mathcal{P}_n$ then $T(\pi) \in \mathcal{P}_n$. If we start with a partition $\pi \in \mathcal{P}_n$ and look at the iterates of T , we get the orbit,

$$O_T(\pi) = \{\pi, T(\pi), T^2(\pi), T^3(\pi), \dots\}.$$

For a fixed value n , the number of partitions of n is finite. Consequently the orbit $O_T(\pi)$ is closed and as a dynamical system on a finite set, T applied to π for any n must eventually form cycles: the partitions $\pi, T(\pi), T^2(\pi), T^3(\pi), \dots$ cannot all be distinct. A partition π is said to be *periodic* under a transformation T if there exists a *positive integer* ℓ such that

$$T^\ell(\pi) = \pi.$$

The smallest such integer ℓ is called the *period* of π under the transformation T .

Example 1. Figure 2 shows the dynamics of T on partitions of 5. The system converges to a cycle of periodic points.

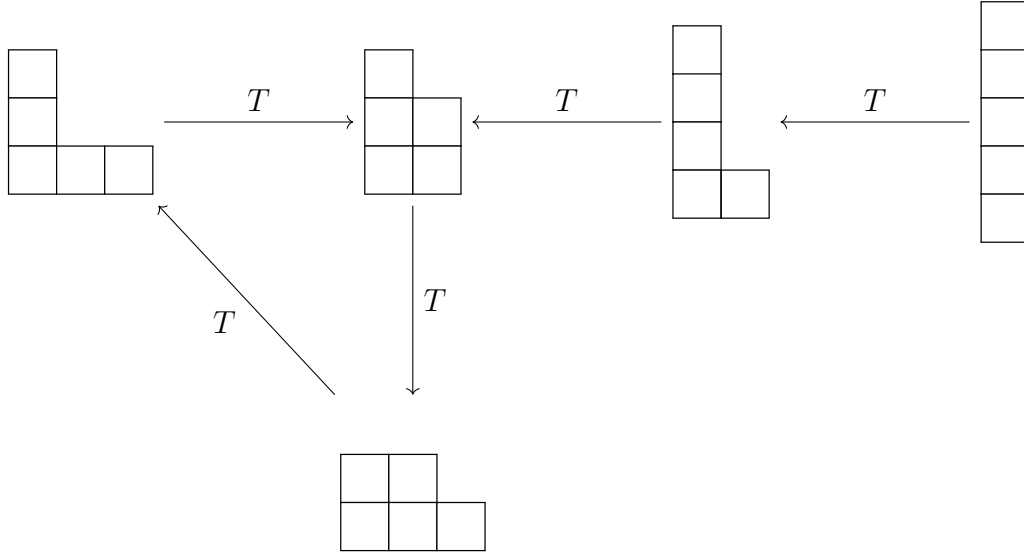


Figure 2: Partitions of 5 under the action T

Thus the partition $\pi = (3, 1, 1) \in \mathcal{P}_5$ has period $\ell = 3$ under the action of T . □

To understand the structure of periodic points, it is useful to examine the distribution of cells along specific diagonals of the Ferrers diagram.

Definition 1 (k^{th} upper diagonal). *For a fixed positive integer k , the k^{th} upper diagonal, denoted by $\text{diag}[k]$, is defined as the set of positions*

$$\text{diag}[k] = \{(1, k), (2, k-1), \dots, (k, 1)\}.$$

A complete characterization of periodic points exists. For any positive integer n , let m be the unique integer such that the m^{th} triangular number $T_m = m(m+1)/2$ satisfies $T_m \leq n < T_{m+1}$. Define $m' = n - T_m$.

The periodic points of T are characterized as follows.

Theorem 1. [10][Bulgarian Solitaire] Let π be a partition of n . Then π is a periodic point of T if and only if

$$\text{Number of cells in } \text{diag}[k] = \begin{cases} k, & \text{when } k = 1, 2, \dots, m, \\ m', & \text{when } k = m + 1, \\ 0, & \text{when } k > m + 1. \end{cases}$$

In the special case where $n = T_m$, under the action T , $\mathcal{P}_n$ contains only one periodic point, namely,

$$(m, m - 1, m - 2, \dots, 2, 1).$$

In the case where n is a non-triangular number, the period length ℓ depends on the initial configuration and is a divisor of $m + 1$. The number of distinct orbits of $\mathcal{P}_n$ under the action of T is equal to the number of necklaces consisting of m' black beads and $(m + 1 - m')$ white beads, where the symmetry group is the cyclic group of order $(m + 1)$. The number of distinct orbits is given by an elegant formula in Theorem 2.

Theorem 2. [4] Given $n \geq 1$, write n uniquely as $T_m + m'$ for some integer m' with $0 \leq m' \leq m$. The number of distinct orbits in $\mathcal{P}_n$ under the operation T is

$$\frac{1}{m + 1} \sum_{d \mid \gcd(m+1, m')} \phi(d) \left(\frac{\frac{m+1}{d}}{\frac{m'}{d}} \right),$$

where $\gcd(m + 1, m')$ denotes the greatest common divisor of $m + 1$ and m' , and $\phi(d)$ is the Euler totient function.

Example 2. Figure 3 shows the orbit of $\pi = (3, 1, 1)$ under the action T . As stated in Theorem 1, π is a periodic point of T . Applying T amounts to shifting cells in the $\text{diag}[m + 1]$ cyclically down the diagonal.

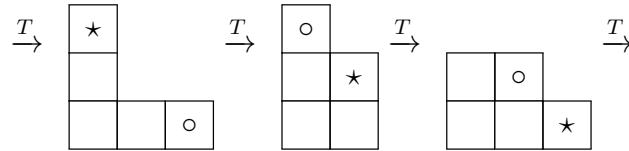


Figure 3: Orbit of $\pi = (3, 1, 1)$ under the action T

□

This orbit corresponds to the unique necklace with $m' = 2$ black beads and $m + 1 - m' = 1$ white bead, shown in Figure 4.

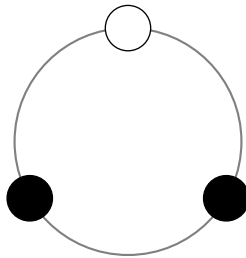


Figure 4: Necklace with 2 black beads and 1 white bead

3 Generalizing to partitions with distinguishable parts

Let π be a periodic point of T as described in Theorem 1. From now on we assume that the cells in the Ferrers diagram of π are distinguishable, and we denote π as,

$$\pi = ((\pi_{11}, \pi_{12}, \pi_{13}, \dots), (\pi_{21}, \pi_{22}, \pi_{23}, \dots), (\pi_{31}, \pi_{32}, \pi_{33}, \dots), \dots)$$

where each $\pi_i = (\pi_{i1}, \pi_{i2}, \pi_{i3}, \dots)$ represents a distinguishable part, which we draw as a column of the Ferrers diagram and π_{ij} is a labeled cell in the i^{th} column and j^{th} row. We denote the set of all partitions with distinguishable parts by $\mathcal{P}^*$, and the subset of all partitions with distinguishable parts of n by $\mathcal{P}_n^*$. The set of all labeled cells of a partition's Ferrers diagram is denoted by $\mathcal{L}_\pi$.

Example 3. For instance, for the partition with distinguishable parts, $\pi^* = ((a, b, c), (d), (e)) \in \mathcal{P}_5^*$, in reduced form, corresponds to the Ferrers diagram in Figure 5.

c		
b		
a	d	e

Figure 5: Ferrers diagram of $\pi^* = ((a, b, c), (d), (e))$

and also we have $\mathcal{L}_{\pi^*} = \{a, b, c, d, e\}$.

□

The size of a part π_i is defined as the number of elements in its tuple and it is equivalent to the column height in the Ferrers diagram and denoted as $|\pi_i|$ and $\sum_{i=1}^{\infty} |\pi_i| = n$.

Inspired by how the cells were interpreted to move in Figure 3, we introduce actions $T_1, \dots, T_4$ on $\mathcal{P}_5^*$, which are ways to remove a distinguishable cell from each column of the Ferrers diagram such that they will be periodic according to Theorem 1. There are two natural ways that we can remove a cell from each column:

1. Remove the lowest cell, π_{i1} for $i = 1, \dots, m$ from each column.
2. Remove the highest cell, π_{in_i} for $i = 1, \dots, m$ from each column.

Moreover, there are two natural ways to place these cells in the new column:

1. Removed cells from columns $i = 1, \dots, m$, place them in the order bottom to top in the new first column.
2. Removed cells from columns $i = 1, \dots, m$, place them in the order top to bottom in the new first column.

Thus, we can introduce four actions by the order in which we place removed cells in the first column of the Ferrers diagram considering different combinations of removal and replacement of the cells.

3.1 The action T_1

Let $\pi \in \mathcal{P}_n^*$ be in reduced form with positive integers $m, n_1, \dots, n_m$, and

$$\pi = ((\pi_{11}, \pi_{12}, \dots, \pi_{1n_1}), (\pi_{21}, \pi_{22}, \dots, \pi_{2n_2}), \dots, (\pi_{m1}, \pi_{m2}, \dots, \pi_{mn_m})).$$

Define $T_1 : \mathcal{P}_n^* \longrightarrow \mathcal{P}_n^*$ by

$$T_1(\pi) = ((\pi_{11}, \pi_{21}, \dots, \pi_{m1}), (\pi_{12}, \dots, \pi_{1n_1}), \dots, (\pi_{m2}, \dots, \pi_{mn_m})).$$

Using Example 3, $\pi^* = ((a, b, c), (d), (e)) \in \mathcal{P}_5^*$,

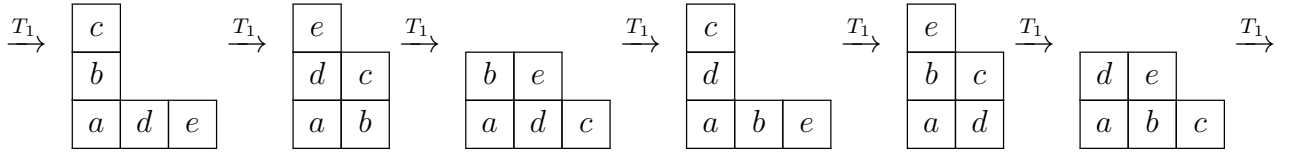


Figure 6: Orbit of π^* under the action T_1

In other words:

1. We remove all of the cells in the first row of the Ferrers diagram.
2. We translate the remaining cells down one square and then one square to the right.
3. We now place the cells that we had removed from the first row into the first column. In doing so, we take care to place the cell that originally was on the $(k, 1)^{\text{th}}$ square onto the $(1, k)^{\text{th}}$ square for $k = 1, \dots, m + 1$.

Suppose we are updating coordinates:

- for $i = 1$, position $(1, j)$ is occupied by the cell that was in position $(j, 1)$.
- for $i > 1$, coordinate (i, j) is occupied by the cell that was in position $(i - 1, j + 1)$ (possibly empty).

3.2 The action T_2

Let $\pi \in \mathcal{P}_n^*$ be in reduced form with positive integers $m, n_1, \dots, n_m$ and

$$\pi = ((\pi_{11}, \pi_{12}, \dots, \pi_{1n_1}), (\pi_{21}, \pi_{22}, \dots, \pi_{2n_2}), \dots, (\pi_{m1}, \pi_{m2}, \dots, \pi_{mn_m})).$$

Define $T_2 : \mathcal{P}_n^* \longrightarrow \mathcal{P}_n^*$ by

$$T_2(\pi) = ((\pi_{m1}, \dots, \pi_{21}, \pi_{11}), (\pi_{12}, \dots, \pi_{1n_1}), \dots, (\pi_{m2}, \dots, \pi_{mn_m})).$$

Using Example 3, $\pi^* = ((a, b, c), (d), (e)) \in \mathcal{P}_5^*$,

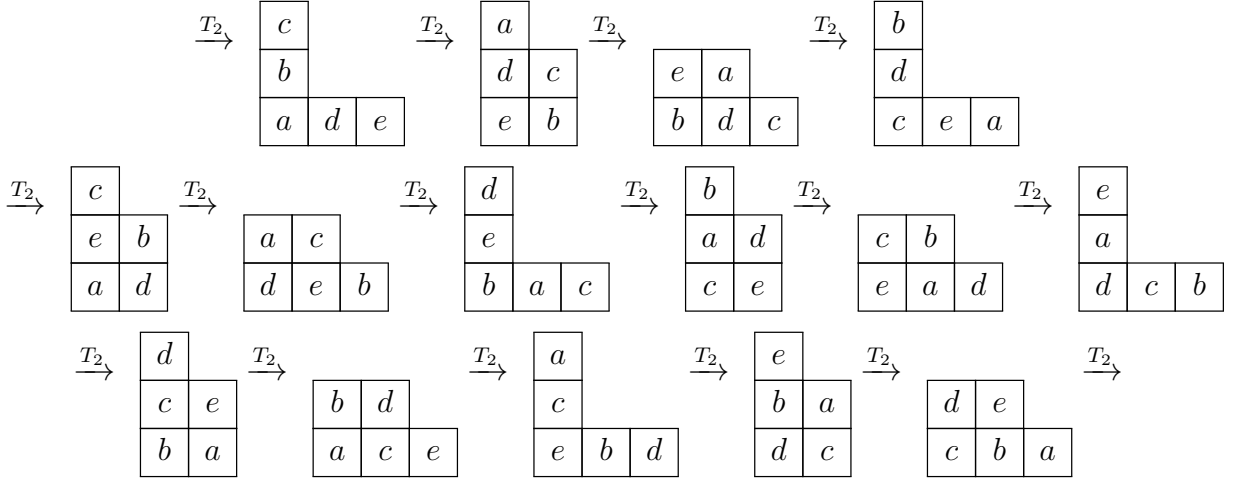


Figure 7: Orbit of π^* under the action T_2

In other words:

1. We remove all of the cells in the first row of the Ferrers diagram.
2. We translate the remaining cells down one square and then one square to the right.
3. We now place the cells that we had removed from the first row into the first column. In doing so, we take care to place the cell that originally was on the leftmost cell of the first row into the highest cell of the first column and the rightmost cell of the first row into the lowest cell of the first column. The cells in between from left to right follow the order top to bottom.

To better describe the movement of individual cells under the action T_2 , we introduce coordinates: for a partition π , let $\pi(i, j)$ denote the cell located in the i^{th} column and j^{th} row of the Ferrers diagram such that $\pi(i, j) = \pi_{ij}$. With this notation, we can express T_2 as follows:

Proposition 1. *Let $\pi \in \mathcal{P}_n^*$ be in reduced form and consider the orbit of π under T_2 . Then*

$$\pi(i, j) = \begin{cases} (T_2\pi)(i+1, j-1) & \text{if } j > 1 \\ (T_2\pi)(1, m+2-i) & \text{if } j = 1 \text{ and } \pi \text{ has } m+1 \text{ cells in bottom row} \\ (T_2\pi)(1, m+1-i) & \text{if } j = 1 \text{ and } \pi \text{ has } m \text{ cells in bottom row.} \end{cases}$$

Remark 1. The rules given in Proposition 1 describe the forward positions of labeled cells under T_2 . Specifically, it expresses where the label originally at position (i, j) in π appears after one application of T_2 . Since we view T_2 as a transformation $T_2 : \mathcal{P}_n^* \rightarrow \mathcal{P}_n^*$, this defines a *left action* on $\mathcal{P}_n^*$.

Example 4. Let $\pi = ((a, b, c), (d), (e)) \in \mathcal{P}_5^*$ as in Figure 7. Then $T_2(\pi)$ is the first tableau in the orbit. Here, $n = 5$, so the largest triangular number less than or equal to 5 is 3, which comes from $m = 2$ since $2(2+1)/2 = 3$. In $T_2(\pi)$, the bottom row has 3 cells, which is $m+1$ with $m = 2$. Therefore, we use the second case of Proposition 1 for cells with $j = 1$. Thus:

- $\pi(1, 1) = a$ should equal $(T_2\pi)(1, m + 2 - 1) = (T_2\pi)(1, 3)$ which is a .
- $\pi(2, 1) = d$ should equal $(T_2\pi)(1, m + 2 - 2) = (T_2\pi)(1, 2)$ which is d .
- $\pi(3, 1) = e$ should equal $(T_2\pi)(1, m + 2 - 3) = (T_2\pi)(1, 1)$ which is e .

For cells with $j > 1$, we use the first case:

- $\pi(1, 2) = b$ should equal $(T_2\pi)(1 + 1, 2 - 1) = (T_2\pi)(2, 1)$ which is b .
- $\pi(1, 3) = c$ should equal $(T_2\pi)(1 + 1, 3 - 1) = (T_2\pi)(2, 2)$ which is c .

This verifies the proposition. □

3.3 The action T_3

Let $\pi \in \mathcal{P}_n^*$ be in reduced form with positive integers $m, n_1, \dots, n_m$, and

$$\pi = ((\pi_{11}, \pi_{12}, \dots, \pi_{1n_1}), (\pi_{21}, \pi_{22}, \dots, \pi_{2n_2}), \dots, (\pi_{m1}, \pi_{m2}, \dots, \pi_{mn_m})).$$

Define $T_3 : \mathcal{P}_n^* \longrightarrow \mathcal{P}_n^*$ by

$$T_3(\pi) = ((\pi_{mn_m}, \dots, \pi_{2n_2}, \pi_{1n_1}), (\pi_{12}, \dots, \pi_{1n_1}), \dots, (\pi_{m2}, \dots, \pi_{mn_m})).$$

Using Example 3, $\pi^* = ((a, b, c), (d), (e)) \in \mathcal{P}_5^*$,

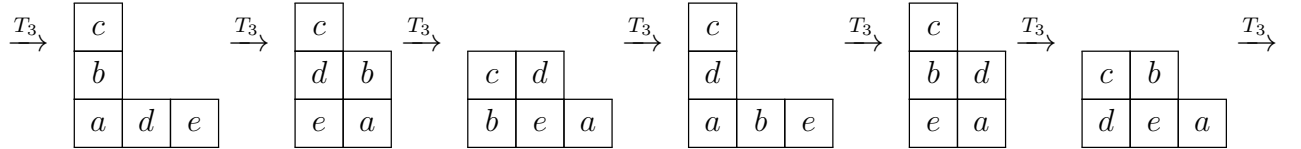


Figure 8: Orbit of π^* under the action T_3

In other words:

1. We remove the highest cell of each column of the Ferrers diagram, F_π .
2. We translate the remaining cells one square to the right creating a new first column.
3. We now place the last cells that we had removed from each column. In doing so, we take care to place removed cells from columns $i = 1, \dots, m$ in the order top to bottom in the new first column.

3.4 The action T_4

Let $\pi \in \mathcal{P}_n^*$ be in reduced form with positive integers $m, n_1, \dots, n_m$, and

$$\pi = ((\pi_{11}, \pi_{12}, \dots, \pi_{1n_1}), (\pi_{21}, \pi_{22}, \dots, \pi_{2n_2}), \dots, (\pi_{m1}, \pi_{m2}, \dots, \pi_{mn_m})).$$

Define $T_4 : \mathcal{P}_n^* \longrightarrow \mathcal{P}_n^*$ by

$$T_4(\pi) = ((\pi_{1n_1}, \pi_{2n_2}, \dots, \pi_{mn_m}), (\pi_{12}, \dots, \pi_{1n_1}), \dots, (\pi_{m2}, \dots, \pi_{mn_m})).$$

Using Example 3, $\pi = ((a, b, c), (d), (e)) \in \mathcal{P}_5^*$,

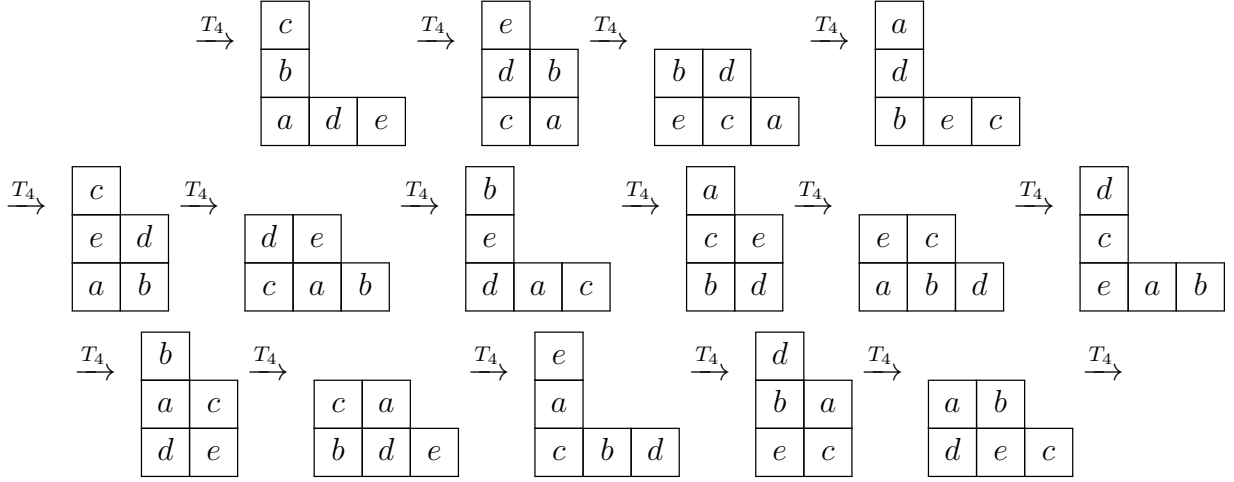


Figure 9: Orbit of π^* under the action T_4

In other words:

1. We remove the highest cell of each column of the Ferrers diagram.
2. We translate the remaining cells one square to the right creating a new first column.
3. We now place the last cells that we had removed from each column. In doing so, we take care to place removed cells from columns $i = 1, \dots, m$ in the order bottom to top in the new first column.

We are trying to figure out the orbit sizes of $\pi \in \mathcal{P}_n^*$ under the actions T_i .

4 The orbits of T_1, T_2, T_3 and T_4

Definition 2 (π^{rev}). Define a map $\text{rev} : \mathcal{P}_n^* \rightarrow \mathcal{P}_n^*$ for some positive integers m, n_m and $\pi \in \mathcal{P}_n^*$. Specifically if

$$\pi = ((\pi_{11}, \pi_{12}, \dots, \pi_{1n_1}), (\pi_{21}, \pi_{22}, \dots, \pi_{2n_2}), \dots, (\pi_{m1}, \pi_{m2}, \dots, \pi_{mn_m})).$$

then

$$\text{rev}(\pi) = ((\pi_{1n_1}, \dots, \pi_{12}, \pi_{11}), (\pi_{2n_2}, \dots, \pi_{22}, \pi_{21}), \dots, (\pi_{mn_m}, \dots, \pi_{m2}, \pi_{m1})),$$

and we denote $\text{rev}(\pi)$ as π^{rev} .

Proposition 2. Let $\pi \in \mathcal{P}_n^*$. The following diagrams commute:

$$\begin{array}{ccc} \pi & \xrightarrow{T_1} & T_1(\pi) \\ \text{rev} \downarrow & & \downarrow \text{rev} \\ \pi^{\text{rev}} & \xrightarrow{T_3} & T_3(\pi^{\text{rev}}) \end{array} \quad \begin{array}{ccc} \pi & \xrightarrow{T_2} & T_2(\pi) \\ \text{rev} \downarrow & & \downarrow \text{rev} \\ \pi^{\text{rev}} & \xrightarrow{T_4} & T_4(\pi^{\text{rev}}) \end{array}$$

i.e., $\text{rev} \circ T_1 = T_3 \circ \text{rev}$ and $\text{rev} \circ T_2 = T_4 \circ \text{rev}$.

Proof. Let $\pi \in \mathcal{P}_n^*$ and for some positive integers m, n_m ,

$$\pi = ((\pi_{11}, \pi_{12}, \dots, \pi_{1n_1}), (\pi_{21}, \pi_{22}, \dots, \pi_{2n_2}), \dots, (\pi_{m1}, \pi_{m2}, \dots, \pi_{mn_m})).$$

By reversing the order of the partition π , we are interchanging the lowest cell π_{i1} with the highest cell π_{in_i} in each column $1, \dots, m$. In actions T_1 and T_2 , we are removing the lowest cell π_{i1} and in actions T_3 and T_4 , we are removing the highest cell π_{in_i} . Since we have reversed the order of π in π^{rev} and when we put back removed cells from each column we follow the opposite order in actions T_1 and T_3 , thus we have $\text{rev} \circ T_1(\pi) = T_3 \circ \text{rev}(\pi)$ for all $\pi \in \mathcal{P}_n^*$, and also we follow the opposite order in actions T_2 and T_4 , thus we have $\text{rev} \circ T_2(\pi) = T_4 \circ \text{rev}(\pi)$ for all $\pi \in \mathcal{P}_n^*$. ■

Corollary 1. $|O_{T_1}(\pi)| = |O_{T_3}(\pi)|$ and $|O_{T_2}(\pi)| = |O_{T_4}(\pi)|$ for all $\pi \in \mathcal{P}_n^*$.

Thus, we only need to study the orbit sizes of $\pi \in \mathcal{P}_n^*$ under the actions T_1 and T_2 .

4.1 The orbit size of π under the action T_1

Theorem 3 (Orbit size of π under T_1). *Let π be in $\mathcal{P}_n^*$. Then*

$$|O_{T_1}(\pi)| = \text{lcm}(1, 2, \dots, m+1) = |O_{T_3}(\pi)|.$$

Proof. We can observe that the coordinates of cells move as follows. Suppose we are updating coordinates:

- for $i = 1$, position $(1, j)$ is occupied by the cell that was in position $(j, 1)$ for $j = 1, \dots, m+1$.
- for $i > 1$, position (i, j) is occupied by the cell that was in position $(i-1, j+1)$ for $i+j \leq m+2$.

Each cell in $\text{diag}[k]$ cycles through the k^{th} diagonal and comes to its original position after k iterations of T_1 . The least common multiple of the integers from 1 to $m+1$; $\text{lcm}(1, 2, \dots, m+1)$ gives us the full period for all cells. ■

4.2 The orbit size of π under the action T_2

4.2.1 σ_π under the action T_2

Let $\pi \in \mathcal{P}_n^*$ be a labeled set partition in reduced form. The action of T_2 on π rearranges the cells of its Ferrers diagram, thereby inducing a permutation on the labels. Since the orbit sizes of π under T are all divisors of $m+1$, we map the cells of the Ferrers diagram of π to the cells of the Ferrers diagram of $T_2^{m+1}(\pi)$, the partition of the same shape as the initial partition π .

Definition 3. *Let $\pi \in \mathcal{P}_n^*$ be in reduced form. Define $\sigma_\pi : \mathcal{L}_\pi \longrightarrow \mathcal{L}_\pi$ by*

$$\sigma_\pi(\pi(i, j)) = (T_2^{m+1}\pi)(i, j).$$

Example 5. Consider $\pi = ((a, b, c), (d), (e))$ as in Example 3. Using Proposition 1 and by Definition 3, we compute:

$$\begin{aligned}\sigma_\pi(a) &= \sigma_\pi(\pi(1, 1)) = (T_2^3\pi)(1, 1) = c, \\ \sigma_\pi(b) &= \sigma_\pi(\pi(1, 2)) = (T_2^3\pi)(1, 2) = d, \\ \sigma_\pi(c) &= \sigma_\pi(\pi(1, 3)) = (T_2^3\pi)(1, 3) = b, \\ \sigma_\pi(d) &= \sigma_\pi(\pi(2, 1)) = (T_2^3\pi)(2, 1) = e, \\ \sigma_\pi(e) &= \sigma_\pi(\pi(3, 1)) = (T_2^3\pi)(3, 1) = a.\end{aligned}$$

This gives the permutation $\sigma_\pi = (a c b d e)$. □

Definition 4 (Order of a permutation). *For a permutation σ in the symmetric group S_n , the order of σ , denoted $\text{ord}(\sigma)$, is the smallest positive integer m such that $\sigma^m = \text{id}$, where id is the identity permutation.*

We can find $\text{ord}(\sigma)$ by taking the least common multiple of the lengths of the disjoint cycles in the cycle decomposition of σ . The orbit of a partition $\pi \in \mathcal{P}_n^*$ under the action T_2 is characterized by a dual periodicity: the Ferrers diagram shape returns to its original form every $m + 1$ iterations of T , while the permutation σ_π governs the cyclic rearrangement of the labels within this fixed shape. The full orbit under T_2 is only completed after $\text{ord}(\sigma_\pi)$ such shape cycles, resulting in an orbit size of $|O_{T_2}(\pi)|$, which is a divisor of $(m + 1) \cdot \text{ord}(\sigma_\pi)$. Therefore, we have the following theorem.

Theorem 4 (Orbit size of π under the action T_2). *The orbit size of a periodic point π under the action T_2 is a divisor of $(m + 1) \cdot \text{ord}(\sigma_\pi)$.*

Example 6 (Orbit sizes for $\pi = ((a, b, c), (d), (e))$). Consider the colored partition $\pi = ((a, b, c), (d), (e)) \in \mathcal{P}_5^*$ with $m = 2$ and $m' = 2$. The underlying uncolored partition has period $m + 1 = 3$ under the classical Bulgarian solitaire operation T . For the colored generalizations, we compute the orbit sizes under action T_2 as follows:

The permutation induced on labels is $\sigma_\pi = (a c b d e)$, a 5-cycle with $\text{ord}(\sigma_\pi) = 5$. The orbit size is

$$|O_{T_2}(\pi)| = 15 = 3 \cdot 5 = (m + 1) \cdot \text{ord}(\sigma_\pi).$$

This matches the explicit orbit shown in Figure 7. The orbit size equals $(m + 1) \cdot \text{ord}(\sigma_\pi)$, which is indeed a divisor of $(m + 1) \cdot \text{ord}(\sigma_\pi)$ (since every number divides itself). This illustrates the theorem, though it should be noted that in general, the orbit size may be a proper divisor of $(m + 1) \cdot \text{ord}(\sigma_\pi)$ when additional symmetries are present. □

4.3 Algorithm for determining the cycles of σ_π under the action T_2

We now focus on describing the order of σ_π . We start by defining some notation that will be useful.

Definition 5 (Extra cells). *Let $\pi \in \mathcal{P}_n^*$ be a periodic point. The cells located on the $(m + 1)^{\text{th}}$ diagonal, denoted $\text{diag}[m + 1]$, are called extra cells. The number of extra cells is given by $m' = n - \frac{m(m+1)}{2}$, where $0 \leq m' \leq m$.*

Definition 6 (Row and column classification). *Let $\pi \in \mathcal{P}_n^*$ be a periodic point with m' extra cells.*

- *Define h_i as the height of the i^{th} highest row with an extra cell, where $(m+2-h_i, h_i)$ is the extra cell and $h_1 > h_2 > \dots > h_{m'}$.*
- *Define $l_i = h_{m'+1-i}$ as the height of the i^{th} lowest row with an extra cell, where $(m+2-l_i, l_i)$ is the extra cell and $l_1 < l_2 < \dots < l_{m'}$.*

Definition 7 (Long row, long column, short row, short column). *Let $\pi \in \mathcal{P}_n^*$ be a periodic point with m' extra cells.*

- *The rows that contain an extra cell are called long rows, and the columns that contain an extra cell are called long columns.*
- *The rows that do not contain an extra cell are called short rows, and the columns that do not contain an extra cell are called short columns.*

4.3.1 Algorithm for coloring cells

We will show that the cycles of σ_π are related to a certain coloring of π . We color cells in the following way.

Let $I = \{i_1, i_2, \dots, i_{m'}\}$ be the set of long column locations with elements ordered such that $i_1 < i_2 < \dots < i_{m'}$. This means (i_k, h_k) for $1 \leq k \leq m'$ are the extra cells in π . Let

$$E = \left((i_1, h_1), \dots, (i_{m'}, h_{m'}) \right)$$

be an ordered list containing coordinates of consecutive m' extra cells in $\text{diag}[m+1]$. Define a function $X : \mathbb{N}^2 \times \mathbb{N}^2 \rightarrow \mathbb{N}^2$ by

$$X\left((a_1, b_1), (a_2, b_2)\right) = (a_1, b_2),$$

which gives the intersection cell coordinate of a column containing cell (a_1, b_1) and a row containing cell (a_2, b_2) . Apply X to each consecutive pair of the set E , and denote the new set as $X(E)$:

$$X^0(E) = E \quad \text{and} \quad X(E) = \left((i_1, h_2), \dots, (i_{m'-1}, h_{m'}) \right).$$

We can apply X to E , k times for $1 \leq k \leq m' - 1$; to get

$$X^k = X(X^{k-1}(E)).$$

In particular, $X^{m'-1}(E) = \left((i_1, h_{m'}) \right)$.

Now we color the cells of π with distinct colors $\{C_1, \dots, C_{\lceil \frac{m'}{2} \rceil}\}$ in two stages as follows.

First, color $E \cup X^{m'-1}(E)$ with color C_1 , $X(E) \cup X^{m'-2}(E)$ with color C_2 , $X^2(E) \cup X^{m'-3}(E)$ with color C_3 , ..., and $X^{\lceil \frac{m'}{2} \rceil - 1}(E) \cup X^{m' - \lceil \frac{m'}{2} \rceil}(E)$ with color $C_{\lceil \frac{m'}{2} \rceil}$. Then starting from the cells colored C_i , we color downward in the same column and leftward in the same row with C_i , until we meet a cell that has already been colored.

Algorithm 1 Cell Coloring Procedure

```
1: Given: Diagonal  $\text{diag}[m + 1]$  with  $m'$  extra cells ( $m' \geq 1$ )
2:
3: Define ordered set  $E$  along diagonal:
4:    $E \leftarrow ((i_1, h_1), \dots, (i_{m'}, h_{m'}))$ 
5:
6: Compute iterated intersections:
7: for  $k \leftarrow 1$  to  $m' - 1$  do
8:    $X^0(E) = E$ 
9:    $X^k(E)$ 
10: end for
11: (Where  $X((a_1, b_1), (a_2, b_2)) = (a_1, b_2)$ )
12:
13: Color pattern initialization:
14: Number of colors needed:  $m'/2$ 
15: Assign distinct colors  $\{C_1, \dots, C_{\lceil m'/2 \rceil}\}$ 
16:
17: Apply coloring rules:
18: for  $t \leftarrow 0$  to  $\lceil \frac{m'}{2} \rceil - 1$  do
19:   Color  $X^t(E) \cup X^{m'-1-t}(E)$  with  $C_{t+1}$ 
20: end for
21:
22: Propagate colors through grid:
23: for all initially colored cell  $(a, b)$  with color  $C_i$  do
24:   Downward: Color with  $C_i$  all cells below  $(a, b)$  until reaching another colored cell
    or the bottom of the column.
25:   Leftward: Color with  $C_i$  all cells left of  $(a, b)$ , until reaching a colored cell or the
    end of the row.
26: end for
```

Example 7. Let us examine the colored partition in Figure 10 of $n = 31$ with $m = 7$ and $m' = 3$. The diagonal $\text{diag}[8]$ contains $m' = 3$ extra cells at positions $(1, 8)$, $(4, 5)$, and $(7, 2)$, where coordinates (i, j) denote column i and row j (with row 1 being the bottom).

1. Initial Setup:

- $E = ((1, 8), (4, 5), (7, 2))$ (ordered by increasing column index)
- Since $m' = 3$ is odd, we need $\lceil 3/2 \rceil = 2$ colors: Yellow (C_1) and Green (C_2)

2. Intersection Computation:

$$X^1(E) = ((1, 5), (4, 2))$$
$$X^2(E) = ((1, 2))$$

3. Initial Coloring:

- C_1 (Yellow): $X^0(E) \cup X^2(E) = ((1, 8), (4, 5), (7, 2), (1, 2))$

- C_2 (Green): $X^1(E) \cup X^1(E) = ((1, 5), (4, 2))$

4. Color Propagation:

- From (1, 8) (Yellow): Propagate downward until (1, 5). The cells (1, 7), (1, 6) get colored yellow.
- From (4, 5) (Yellow): Propagate downward until (4, 2) and leftward until (1, 5). The cells (4, 4), (4, 3), (3, 5), (2, 5) get colored.
- From (7, 2) (Yellow): Propagate downward to the bottom of the column and leftward until (4, 2). The cells (7, 1) (6, 2), (5, 2) get colored.
- From (1, 2) (Yellow): Propagate downward until (1, 2). The cells (1, 1) get colored.
- From (1, 5) (Green): Propagate downward until (1, 2). The cells (1, 4), (1, 3) get colored.
- From (4, 2) (Green): Propagate downward to the bottom of the column and leftward until (1, 2). The cells (4, 1), (3, 2), (2, 2) get colored.

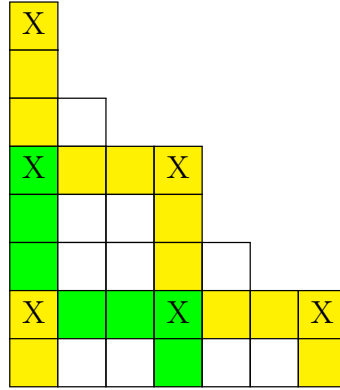


Figure 10: The cell coloring procedure applied to a $\pi \in \mathcal{P}_{31}^*$. The cells marked with ‘X’ indicate the initially colored intersection points before propagation.

□

Example 8. Let us examine the colored partition in Figure 11 of $n = 86$ with $m = 12$ and $m' = 8$. The diagonal $\text{diag}[13]$ contains $m' = 8$ extra cells at positions (1, 13), (2, 12), (3, 11), (4, 10), (5, 9), (6, 8), (9, 5), and (12, 2).

1. Initial Setup:

- $E = ((1, 13), (2, 12), (3, 11), (4, 10), (5, 9), (6, 8), (9, 5), (12, 2))$ (ordered by increasing column index)
- Since $m' = 8$ is even, we need $8/2 = 4$ colors: Yellow (C_1), Green (C_2), Orange (C_3), Cyan (C_4)

2. Intersection Computation:

$$X^1(E) = \left((1, 12), (2, 11), (3, 10), (4, 9), (5, 8), (6, 5), (9, 2) \right)$$

$$X^2(E) = \left((1, 11), (2, 10), (3, 9), (4, 8), (5, 5), (6, 2) \right)$$

$$X^3(E) = \left((1, 10), (2, 9), (3, 8), (4, 5), (5, 2) \right)$$

$$X^4(E) = \left((1, 9), (2, 8), (3, 5), (4, 2) \right)$$

$$X^5(E) = \left((1, 8), (2, 5), (3, 2) \right)$$

$$X^6(E) = \left((1, 5), (2, 2) \right)$$

$$X^7(E) = \left((1, 2) \right)$$

3. Initial Coloring:

- C_1 (Yellow): $X^0(E) \cup X^7(E) = E \cup \left((1, 2) \right)$
- C_2 (Green): $X^1(E) \cup X^6(E)$
- C_3 (Orange): $X^2(E) \cup X^5(E)$
- C_4 (Cyan): $X^3(E) \cup X^4(E)$

4. Color Propagation:

- Note that the top 5 rows (rows 13, 12, 11, 10, 9) consist entirely of intersection points and are already colored by the initial coloring procedure. Propagation begins from the 6th row from the top (row 8) downward.
- **Row 8 Propagation:**
 - From (6, 8) (Yellow): Propagate downward to (6, 7), (6, 6) until (6, 5) (Green)
 - From (1, 8) (Orange): Propagate downward to (1, 7), (1, 6) until (1, 5) (Green)
 - From (2, 8) (Cyan): Propagate downward to (2, 7), (2, 6) until (2, 5) (Orange)
 - From (3, 8) (Cyan): Propagate downward to (3, 7), (3, 6) until (3, 5) (Cyan)
 - From (4, 8) (Orange): Propagate downward to (4, 7), (4, 6) until (4, 5) (Cyan)
 - From (5, 8) (Green): Propagate downward to (5, 7), (5, 6) until (5, 5) (Orange)
- **Row 5 Propagation:**
 - From (6, 5) (Green): Propagate downward to (6, 4), (6, 3) until (6, 2) (Orange)
 - From (9, 5) (Yellow): Propagate downward to (9, 4), (9, 3) until (9, 2) (Green); leftward to (8, 5), (7, 5) until (6, 5) (Green)
 - From (1, 5) (Green): Propagate downward to (1, 4), (1, 3) until (1, 2) (Yellow)

- From (2, 5) (Orange): Propagate downward to (2, 4), (2, 3) until (2, 2) (Green)
- From (3, 5) (Cyan): Propagate downward to (3, 4), (3, 3) until (3, 2) (Orange)
- From (4, 5) (Cyan): Propagate downward to (4, 4), (4, 3) until (4, 2) (Cyan)
- From (5, 5) (Orange): Propagate downward to (5, 4), (5, 3) until (5, 2) (Cyan)
- **Row 2 Propagation:**
 - From (12, 2) (Yellow): Propagate downward to (12, 1); leftward to (11, 2), (10, 2) until (9, 2) (Green)
 - From (9, 2) (Green): Propagate downward to (9, 1); leftward to (8, 2), (7, 2) until (6, 2) (Orange)
 - From (1, 2) (Yellow): Propagate downward to (1, 1)
 - From (2, 2) (Green): Propagate downward to (2, 1)
 - From (3, 2) (Orange): Propagate downward to (3, 1)
 - From (4, 2) (Cyan): Propagate downward to (4, 1)
 - From (5, 2) (Cyan): Propagate downward to (5, 1)
 - From (6, 2) (Orange): Propagate downward to (6, 1)

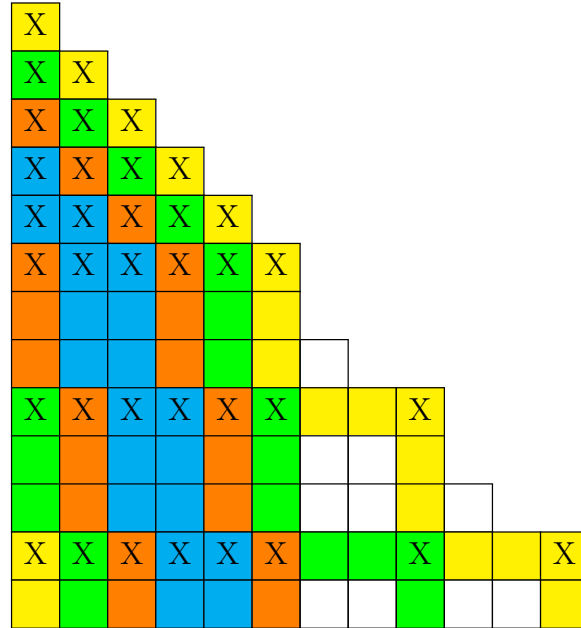


Figure 11: The cell coloring procedure applied to a $\pi \in \mathcal{P}_{86}^*$. The cells marked with 'X' indicate the initially colored intersection points before propagation.

□

Remark 2. Not every cell of π gets colored by the coloring Algorithm 1. A direct consequence of how the coloring works is that the cells that do not get colored are precisely those that are in a short row and a short column of $\pi \in \mathcal{P}^*$. This can be observed in Figure 11, where the white cells are exactly those that lie in rows and columns that do not contain any extra cells.

Moreover, the initial coloring assignment in Algorithm 1 pairs sets $X^t(E)$ and $X^{m'-1-t}(E)$ for $t = 0, \dots, \lceil m'/2 \rceil - 1$ from which the colors propagate. When m' is even, each color is assigned to two disjoint sets (so it has two colored ‘components’ like in Figure 10) and when m' is odd, the last color (when $t = \lfloor m'/2 \rfloor$) is assigned to only one set (since $X^t(E) = X^{m'-1-t}(E)$). This is illustrated in Figure 10 for the case $m' = 3$ (odd), where the green color (last color) has only one colored component.

4.3.2 From colored partitions to permutation cycles

We now describe the connection between the colored components of π and the structure of σ_π induced by the T_2 action. The coloring algorithm not only partitions the Ferrers diagram visually but also reveals the underlying cycle decomposition of σ_π . We will show that each colored region corresponds to an independent cycle in σ_π . We do this by defining a permutation σ_π^{col} based on the coloring and showing that $\sigma_\pi^{col} = \sigma_\pi$.

4.3.3 σ_π^{col} of a colored partition

We define a permutation σ_π^{col} based on the coloring of π produced by the cell coloring procedure by specifying the cycles of σ_π . We determine “colored” cycles, $\sigma_\pi^{C_i}, 1 \leq i \leq \lceil \frac{m'}{2} \rceil$ in the following way. Assume cycles $\sigma_\pi^{C_1}, \sigma_\pi^{C_2}, \dots, \sigma_\pi^{C_{i-1}}$ have been determined, the i^{th} cycle $\sigma_\pi^{C_i}$ is determined as follows.

Initialize: We start the cycle at the label in position $(1, h_i)$ in the first column which is colored in C_i .

Tracing the first vcomponent of color C_i : We map the first label $\pi(1, h_i)$ to the next connected label with the same color weakly southeast of it and we continue this process until we reach the most south east cell with color C_i .

Note that when m' is even, each color is assigned to two disjoint sets (so two components) and when m' is odd, the last color has only one component. When m' is odd and $i = \lceil m'/2 \rceil$, the cycle terminates at position $(m + 2 - h_i, 1)$, otherwise we jump to the position $(1, l_i)$.

Tracing the second component of color C_i : We map the first label $\pi(1, l_i)$ to the next connected label with the same color weakly southeast of it and we continue this process until we reach the most southeast cell with color C_i .

Termination: A cycle $\sigma_\pi^{C_i}$ starts with coordinate $(1, h_i)$ and is complete when it reaches the coordinate $(m + 2 - h_i, 1)$.

Definition 8. Let $\pi \in \mathcal{P}_n^*$ be a colored partition with label set $\mathcal{L}_\pi$. The colored permutation $\sigma_\pi^{col} : \mathcal{L}_\pi \rightarrow \mathcal{L}_\pi$ is defined as the product

$$\sigma_\pi^{col} = \sigma_\pi^{C_1} \sigma_\pi^{C_2} \dots \sigma_\pi^{C_{\lceil m'/2 \rceil}},$$

where the $\sigma_\pi^{C_i}$ are the disjoint cycles corresponding to the cells colored C_i for $1 \leq i \leq \lceil m'/2 \rceil$.

Remark 3 (Movement types of σ_π^{col}). For any cell position (p, q) , the pre-image under σ_π^{col} is determined by the following cases:

1. **Fixed points (uncolored cells)**, Row q is short and Column p is short:

$$\sigma_\pi^{col}(\pi(p, q)) = \pi(p, q).$$

2. **Downward $\downarrow$ move**, Row q is short and Column p is long:

$$\sigma_\pi^{col}(\pi(p, q + 1)) = \pi(p, q).$$

3. **Diagonal $\searrow$ move**, Row q is long and Column $p - 1$ is long ($p \geq 2$):

$$\sigma_\pi^{col}(\pi(p - 1, q + 1)) = \pi(p, q).$$

4. **Rightward $\rightarrow$ move**, Row q is long and Column $p - 1$ is short ($p \geq 2$):

$$\sigma_\pi^{col}(\pi(p - 1, q)) = \pi(p, q).$$

5. **Jump $\circlearrowleft$ move**, Row q is long and Column $p = 1$. Then $q = l_i$ or $q = h_i$:

$$\sigma_\pi^{col}(\pi(m + 2 - l_i, 1)) = \pi(1, l_i).$$

$$\sigma_\pi^{col}(\pi(m + 2 - h_i, 1)) = \pi(1, h_i).$$

Example 9. We demonstrate the cycle tracing procedure using the colored partition in Figure 10, where $n = 31$, $m = 7$, and $m' = 3$. The colored cells form two color classes: Yellow (C_1) and Green (C_2). We trace the cycles for each color following the procedure described in Definition 8.

Yellow Cycle (C_1):

- **First Component:** We start at position $(1, h_1) = (1, 8)$ with label $\pi(1, 8)$. Following the transition rules:
 - $(1, 8) \downarrow (1, 7)$: Downward move (Row 7 is short, Column 1 is long)
 - $(1, 7) \downarrow (1, 6)$: Downward move (Row 6 is short, Column 1 is long)
 - $(1, 6) \searrow (2, 5)$: Diagonal move (Row 5 is long, Column 1 is long)
 - $(2, 5) \rightarrow (3, 5)$: Rightward move (Row 5 is long, Column 2 is short)
 - $(3, 5) \rightarrow (4, 5)$: Rightward move (Row 5 is long, Column 3 is short)
 - $(4, 5) \downarrow (4, 4)$: Downward move (Row 4 is short, Column 4 is long)
 - $(4, 4) \downarrow (4, 3)$: Downward move (Row 3 is short, Column 3 is long)
 - $(4, 3) \searrow (5, 2)$: Diagonal move (Row 2 is short, Column 4 is long)
 - $(5, 2) \rightarrow (6, 2)$: Rightward move (Row 2 is long, Column 5 is short)
 - $(6, 2) \rightarrow (7, 2)$: Rightward move (Row 2 is long, Column 6 is short)
 - $(7, 2) \downarrow (7, 1)$: Downward move (Row 7 is short, Column 1 is long)
 - $(7, 1) \circlearrowleft (1, 2)$: Jump move ($p = 1$ and $q = l_1 = 2$)

• **Second Component:** We continue from $(1, l_1) = (1, 2)$:

- $(1, 2) \downarrow (1, 1)$: Downward move (Row 1 is short, Column 1 is long)
- $(1, 1) \circlearrowleft (1, 8)$: Jump move ($p = 1$ and $q = h_1 = 8$) (from bottom row back to starting position)

Thus the yellow cycle is: $(1, 8) \downarrow (1, 7) \downarrow (1, 6) \searrow (2, 5) \rightarrow (3, 5) \rightarrow (4, 5) \downarrow (4, 4) \downarrow (4, 3) \searrow (5, 2) \rightarrow (6, 2) \rightarrow (7, 2) \downarrow (7, 1) \circlearrowleft (1, 2) \downarrow (1, 1) \circlearrowleft (1, 8)$.

Green Cycle (C_2):

• **First Component:** We start at position $(1, h_2) = (1, 5)$ with label $\pi(1, 5)$. Following the transition rules:

- $(1, 5) \downarrow (1, 4)$: Downward move (Row 4 is short, Column 1 is long)
- $(1, 4) \downarrow (1, 3)$: Downward move (Row 3 is short, Column 1 is long)
- $(1, 3) \searrow (2, 2)$: Diagonal move (Row 2 is long, Column 1 is long)
- $(2, 2) \rightarrow (3, 2)$: Rightward move (Row 2 is long, Column 2 is short)
- $(3, 2) \rightarrow (4, 2)$: Rightward move (Row 2 is long, Column 3 is short)
- $(4, 2) \downarrow (4, 1)$: Downward move (Row 1 is short, Column 4 is long)
- $(4, 1) \circlearrowleft (1, 5)$: Jump move ($p = 1$ and $q = l_2 = 5$) (from bottom row back to starting position)

Thus the green cycle is: $(1, 5) \downarrow (1, 4) \downarrow (1, 3) \searrow (2, 2) \rightarrow (3, 2) \rightarrow (4, 2) \downarrow (4, 1) \circlearrowleft (1, 5)$.

The permutation σ_π^{col} on the colored cells consists of these two disjoint cycles: the yellow cycle of length 14 and the green cycle of length 7.

The uncolored cells (white cells in Figure 10) are fixed points of σ_π^{col} .

□

We advertised that we will prove $\sigma_\pi^{col} = \sigma_\pi$. We first confirm this for our Example 9.

Example 10 (First $m + 1$ iterations of T_2 on $\pi \in \mathcal{P}_{31}^*$). Consider the colored partition $\pi \in \mathcal{P}_{31}^*$ with $m = 7$ and $m' = 3$ as shown in Figure 12(a). The first 8 iterations of T_2 are displayed in Figure 12.

After $m + 1 = 8$ iterations (Figure 12(i)), we can observe the permutation σ_π by comparing the labels with their original positions in Figure 12(a). By Definition 3, $\sigma_\pi(\pi(i, j)) = (T_2^{m+1}\pi)(i, j)$ we compute cycles of σ_π :

$$\sigma_\pi = (1\ 2\ 3\ 6\ 7\ 8\ 12\ 16\ 22\ 23\ 24\ 31\ 18\ 25)\ (5\ 9\ 13\ 19\ 20\ 21\ 28)$$

The white (uncolored) cells remain fixed points under σ_π . This demonstrates that after $m + 1$ iterations of T_2 , we obtain the permutation $\sigma_\pi (= \sigma_\pi^{col})$ as defined in Definition 8, whose cycle structure is determined by the colored regions.

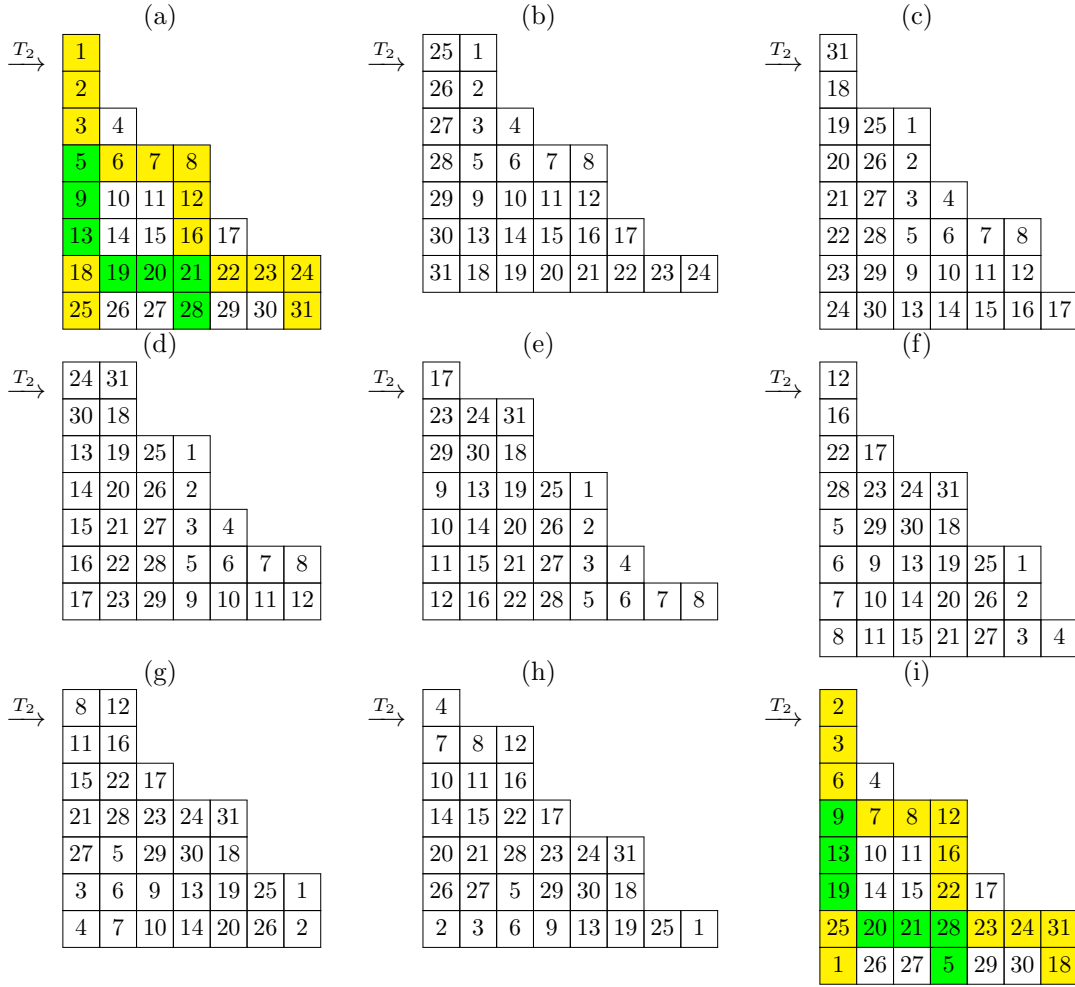


Figure 12: First 8 iterations of T_2 applied to the colored partition $\pi \in \mathcal{P}_{31}^*$ from Figure 10. Tableaux (a) through (i) show consecutive applications of T_2 and Tableau (i) shows the configuration after $m+1 = 8$ iterations, where the permutation σ_π becomes apparent.

□

We now show that σ_π^{col} indeed equals the permutation σ_π induced by T_2^{m+1} . This requires analyzing the behavior of cells under the transformation and verifying that our coloring and cycle-tracing procedures correctly capture the dynamics.

We first prove a lemma that will be useful.

Lemma 1 (Cell movement under T_2^{-1}). *Let $\pi \in \mathcal{P}_n^*$ be a periodic point and consider the inverse transformation T_2^{-1} . Then the following hold for a cell initially at position (p, q) in π :*

- (1) *If (p, q) is in a short row and a short column, then after each application of T_2^{-1} , the cell remains in a short row and a short column.*
- (2) *If (p, q) is in a short row and a long column, then after each application of T_2^{-1} , the cell remains in a short row and a long column until it reaches the bottom row, at which point it jumps to a long row and a short column.*

- (3) If (p, q) is in a long row and Column $p-1$ is long ($p \geq 2$), then after each application of T_2^{-1} , the cell moves to a position that maintains the property of being in a long row until it reaches the bottom row, at which point it jumps to a long row and a long column.
- (4) If (p, q) is in a long row and Column $p-1$ is short ($p \geq 2$), then after each application of T_2^{-1} , the cell moves to a position that maintains the property of being in a long row until it reaches the bottom row, at which point it jumps to a short row and a long column.

Proof. We prove each case separately by tracing the movement of the cell under T_2^{-1} using Proposition 1.

Case 1: (p, q) is in a short row and a short column. Since π is a periodic partition, its Ferrers diagram satisfies the diagonal conditions: there are exactly k cells in diagonal k for $1 \leq k \leq m$, and m' cells in diagonal $m+1$. The condition ensures (p, q) lies in the first m diagonals, where these conditions imply the existence of specific empty spaces:

- In row q , there is an empty space at $(m+2-q, q)$ because the row has length $m+1-q$.
- In column p , there is an empty space at $(p, m+2-p)$ because the column has height $m+1-p$.

Now consider applying T_2^{-1} using Proposition 1:

For $q > 1$, after one application of T_2^{-1} , the cell moves to $(p+1, q-1)$ and remains in a short row and a short column. This is because empty spaces that were in $(m+2-q, q)$ and $(p, m+2-p)$ move to $(m+3-q, q-1)$ and $(p+1, m+1-p)$, respectively.

For $q = 1$, after one application of T_2^{-1} , using the third case of Proposition 1 (for partitions with m cells in the bottom row), sends label $\pi(p, 1)$ to $(1, m+1-p)$. Meanwhile, the empty space from column p , $(p, m+2-p)$ moves to $(p+1, m+1-p)$, remaining in the same row as $(1, m+1-p)$ and also the empty space in position $(m+1, 1)$, will move to $(1, m+1)$. Thus, the label $\pi(p, 1)$ in position $(1, m+1-p)$ remains in a short row and short column.

Case 2: (p, q) is in a short row and a long column. The movement of label $\pi(p, q)$ from (p, q) to $(p+1, q-1)$ preserves the property of being in a short row and a long column until $q = 1$.

When $q = 1$, the cell is in the bottom row and in a long column. Then by Proposition 1 (third case for m cells in bottom row), we have $\pi(i, 1) = (T_2\pi)(1, m+1-i)$. So the cell moves to $(1, m+1-p)$. Since p is a long column, $(p, m+2-p)$ is an extra cell. After one application of T_2^{-1} , the extra cell has position $(p+1, m+1-p)$. So $(1, m+1-p)$ is in a long row and the first column is short since there were only m cells in the bottom row. Thus, the cell changes from (short row, long column) to (long row, short column) after reaching the bottom row.

Case 3: (p, q) is in a long row and Column $p-1$ is long ($p \geq 2$).

Since the column $p-1$ is long, there is an extra cell in position $(p-1, m+3-p)$ and applying T_2^{-1} moves the cell to $(p, m+2-p)$.

For $q > 1$, applying T_2^{-1} moves the cell to $(p+1, q-1)$ by Proposition 1. The status of $(p+1, q-1)$ maintains the property of being in a long row.

For $q = 1$, after one application of T_2^{-1} , using the second case of Proposition 1 (for partitions with $m+1$ cells in the bottom row), sends label $\pi(p, 1)$ to $(1, m+2-p)$. This position is in a long row (since there is an extra cell at position $(p, m+2-p)$) and the first column is long (since there were $m+1$ cells in the bottom row). Thus, the cell changes from (long row, long or short column) to (long row, long column) after reaching the bottom row.

Case 4: (p, q) is in a long row and Column $p-1$ is short ($p \geq 2$). Since the column $p-1$ is short, there is an empty space in position $(p-1, m+3-p)$ and applying T_2^{-1} moves the empty space to $(p, m+2-p)$.

For $q > 1$, applying T_2^{-1} moves the cell to $(p+1, q-1)$ by Proposition 1. The status of $(p+1, q-1)$ maintains the property of being in a long row.

For $q = 1$, after one application of T_2^{-1} , using the second case of Proposition 1 (for partitions with $m+1$ cells in the bottom row), sends label $\pi(p, 1)$ to $(1, m+2-p)$. This position is in a short row (since there is an empty space at position $(p, m+2-p)$) and the first column is long (since there were $m+1$ cells in the bottom row). Thus, the cell changes from (long row, long or short column) to (short row, long column) after reaching the bottom row. ■

Now we are ready to prove the main result in this subsection.

Theorem 5. *The colored permutation σ_π^{col} defined by the cycle tracing algorithm equals the permutation σ_π induced by T_2^{m+1} on the labeled cells:*

$$\sigma_\pi^{col} = \sigma_\pi.$$

Proof. We prove the equivalence by analyzing each movement type in the definition of σ_π^{col} in Remark 3 and showing it corresponds exactly to the action of T_2^{m+1} as given by Proposition 1. For each case, we trace the position of a cell (p, q) through $m+1$ iterations of T_2^{-1} (the inverse of T_2) and show it arrives at the position specified by σ_π^{col} .

- *Case 1: Fixed points (uncolored cells), Row q is short and Column p is short.*

By definition of σ_π^{col} , we have $\sigma_\pi^{col}(\pi(p, q)) = \pi(p, q)$. We need to show that label $\pi(p, q)$ is a fixed point for σ_π , i.e., after $m+1$ iterations of T_2^{-1} , it will be in the original position (p, q) .

Table 1 summarizes the result of applying T_2^{-1} to the label in position (p, q) . We apply Proposition 1 to trace what happens when T_2^{-1} is applied.

Note that by part (1) of Lemma 1, after several applications of T_2^{-1} , when label $\pi(p, q)$ reaches the bottom row, since that row is short, third rule of Proposition 1 is applied.

- Starting at position (p, q) with $q > 1$, we apply the first rule of Proposition 1 ($j > 1$ case) exactly $q-1$ times. Each application moves the label diagonally: $(i, j) \rightarrow (i+1, j-1)$. After $q-1$ iterations, we reach $(p+q-1, 1)$.

- At position $(p + q - 1, 1)$ with $j = 1$, we apply the third rule of Proposition 1 once, since the bottom row is a short row with m cells. This rule says $\pi(i, 1) = (T_2\pi)(1, m + 1 - i)$. Substituting $i = p + q - 1$, we obtain the new position $(1, m + 2 - p - q)$.
- From position $(1, m + 2 - p - q)$ with $j > 1$, we apply the first rule ($j > 1$ case) exactly $m + 1 - p - q$ times. Each iteration moves $(i, j) \rightarrow (i + 1, j - 1)$. After $m + 1 - p - q$ steps, we reach $(m + 2 - p - q, 1)$.
- At position $(m + 2 - p - q, 1)$ with $j = 1$, we apply the third rule once, since the bottom row is a short row with m cells: $\pi(i, 1) = (T_2\pi)(1, m + 1 - i)$. With $i = m + 2 - p - q$, this gives $(1, p + q - 1)$.
- From position $(1, p + q - 1)$ with $j > 1$, we apply the first rule exactly $p - 1$ times. Each application transforms $(i, j) \rightarrow (i + 1, j - 1)$. After $p - 1$ iterations, we return to the original position (p, q) .

Previous position	Number of iterations of T_2^{-1}	New position
(p, q)	$q - 1$	$(p + q - 1, 1)$
$(p + q - 1, 1)$	1	$(1, m + 2 - p - q)$
$(1, m + 2 - p - q)$	$m + 1 - p - q$	$(m + 2 - p - q, 1)$
$(m + 2 - p - q, 1)$	1	$(1, p + q - 1)$
$(1, p + q - 1)$	$p - 1$	(p, q)

Table 1: Position changes for Case 1.

Table 1 demonstrates a complete 5-step cycle that returns to the starting position (p, q) .

Thus, $\pi(p, q) = (T_2^{m+1}\pi)(p, q)$ and using Definition 3, we have

$$\sigma_\pi(\pi(p, q)) = (T_2^{m+1}\pi)(p, q) = \pi(p, q).$$

- *Case 2: Downward $\downarrow$ move, Row q is short and Column p is long.*

By definition of σ_π^{col} , $\sigma_\pi^{col}(\pi(p, q + 1)) = \pi(p, q)$.

Table 2 summarizes the result of applying T_2^{-1} $m + 1$ times to the label in position (p, q) .

The tracing is similar to Case 1 and we present two critical steps that are different.

Note that by part (2) of Lemma 1, after each application of T_2^{-1} , the cell remains in a short row and a long column until it reaches the bottom row for the **first** time, at which point it jumps to a long row and a short column. Thus, in the **second** visit to the bottom row, it will be in a long row.

- At position $(p + q - 1, 1)$ with $j = 1$, this is the **first** visit to the bottom row. Since (p, q) was in a short row, it will remain short and we apply the third rule of Proposition 1 once: $\pi(i, 1) = (T_2\pi)(1, m + 1 - i)$ for partitions with m cells in the bottom row. Substituting $i = p + q - 1$, we obtain the new position $(1, m + 2 - p - q)$.

- At position $(m + 2 - p - q, 1)$ with $j = 1$, this is the **second** visit to the bottom row. According to part (2) of Lemma 1, we now apply the second rule of Proposition 1: $\pi(i, 1) = (T_2\pi)(1, m + 2 - i)$ for partitions with $m + 1$ cells in the bottom row. With $i = m + 2 - p - q$, this gives $(1, p + q)$.

Previous position	Number of iterations of T_2^{-1}	New position
(p, q)	$q - 1$	$(p + q - 1, 1)$
$(p + q - 1, 1)$	1	$(1, m + 2 - p - q)$
$(1, m + 2 - p - q)$	$m + 1 - p - q$	$(m + 2 - p - q, 1)$
$(m + 2 - p - q, 1)$	1	$(1, p + q)$
$(1, p + q)$	$p - 1$	$(p, q + 1)$

Table 2: Position changes for Case 2.

Thus, $\pi(p, q) = (T_2^{m+1}\pi)(p, q + 1)$ and using Definition 3, we have $\sigma_\pi(\pi(p, q + 1) = (T_2^{m+1}\pi)(p, q + 1) = \pi(p, q)$.

- *Case 3: Diagonal $\searrow$ move, Row q is long and Column $p - 1$ is long ($p \geq 2$).*

By definition of σ_π^{col} , we have $\sigma_\pi^{col}(\pi(p - 1, q + 1)) = \pi(p, q)$. Table 2 summarizes the result of applying T_2^{-1} to the label in position (p, q) .

Note that by Lemma 1 case 3, after each application of T_2^{-1} , the cell moves to a position that maintains the property of being in a long row until it reaches the bottom row, at which point it jumps to a long row and a long column. Thus, in the **second** visit to the bottom row, it will be in a long row.

- At position $(p + q - 1, 1)$ with $j = 1$, this is the **first** visit to the bottom row. Since (p, q) was in a long row, it will remain long and we apply the second rule of Proposition 1: $\pi(i, 1) = (T_2\pi)(1, m + 2 - i)$ for partitions with $m + 1$ cells in the bottom row. Substituting $i = p + q - 1$, we obtain the new position $(1, m + 3 - p - q)$.
- At position $(m + 3 - p - q, 1)$ with $j = 1$, this is the **second** visit to the bottom row. According to the Lemma 1 case 3, we again apply the second rule of Proposition 1: $\pi(i, 1) = (T_2\pi)(1, m + 2 - i)$ for partitions with $m + 1$ cells in the bottom row. With $i = m + 3 - p - q$, this gives $(1, p + q - 1)$.

Previous position	Number of iterations of T_2^{-1}	New position
(p, q)	$q - 1$	$(p + q - 1, 1)$
$(p + q - 1, 1)$	1	$(1, m + 3 - p - q)$
$(1, m + 3 - p - q)$	$m + 2 - p - q$	$(m + 3 - p - q, 1)$
$(m + 3 - p - q, 1)$	1	$(1, p + q - 1)$
$(1, p + q - 1)$	$p - 2$	$(p - 1, q + 1)$

Table 3: Position changes for Case 3.

Thus, $\pi(p, q) = (T_2^{m+1}\pi)(p - 1, q + 1)$ and using Definition 3, we have

$$\sigma_\pi(\pi(p - 1, q + 1) = (T_2^{m+1}\pi)(p - 1, q + 1) = \pi(p, q).$$

- *Case 4: Rightward $\rightarrow$ move, Row q is long and Column $p - 1$ is short ($p \geq 2$).*

By definition of σ_π^{col} , we have $\sigma_\pi^{col}(\pi(p - 1, q)) = \pi(p, q)$. Table 4 summarizes the result of applying T_2^{-1} to the label in position (p, q) .

Note that by Lemma 1 case 3, after each application of T_2^{-1} , the cell moves to a position that maintains the property of being in a long row until it reaches the bottom row, at which point it jumps to a short row and a long column. Thus, in the **second** visit to the bottom row, it will be in a short row.

- At position $(p + q - 1, 1)$ with $j = 1$, this is the **first** visit to the bottom row. Since (p, q) was in a long row, it will remain long and we apply the second rule of Proposition 1: $\pi(i, 1) = (T_2\pi)(1, m + 2 - i)$ for partitions with $m + 1$ cells in the bottom row. Substituting $i = p + q - 1$, we obtain the new position $(1, m + 3 - p - q)$.
- At position $(m + 3 - p - q, 1)$ with $j = 1$, this is the **second** visit to the bottom row. According to the Lemma 1 case 4, we apply the third rule of Proposition 1: $\pi(i, 1) = (T_2\pi)(1, m + 1 - i)$ for partitions with m cells in the bottom row. With $i = m + 3 - p - q$, this gives $(1, p + q - 2)$.

Previous position	Number of iterations of T_2^{-1}	New position
(p, q)	$q - 1$	$(p + q - 1, 1)$
$(p + q - 1, 1)$	1	$(1, m + 3 - p - q)$
$(1, m + 3 - p - q)$	$m + 2 - p - q$	$(m + 3 - p - q, 1)$
$(m + 3 - p - q, 1)$	1	$(1, p + q - 2)$
$(1, p + q - 2)$	$p - 2$	$(p - 1, q)$

Table 4: Position changes for Case 4.

Thus, $\pi(p, q) = (T_2^{m+1}\pi)(p - 1, q)$ and using Definition 3, we have

$$\sigma_\pi(\pi(p - 1, q)) = (T_2^{m+1}\pi)(p - 1, q) = \pi(p, q).$$

- *Case 5: Jump $\circ$ move, Row q is long and Column $p = 1$.*

Then $q = l_i$ or $q = h_i$ and by definition of σ_π^{col} ,

$$\sigma_\pi^{col}(\pi(m + 2 - l_i, 1)) = \pi(1, l_i) \quad \text{and} \quad \sigma_\pi^{col}(\pi(m + 2 - h_i, 1)) = \pi(1, h_i).$$

Table 5 summarizes the result of applying T_2^{-1} to the label in position $(1, l_i)$.

- Starting at position $(1, l_i)$ with $l_i > 1$, we apply the first rule of Proposition 1 ($j > 1$ case) exactly $l_i - 1$ times. Each application moves the label diagonally: $(i, j) \rightarrow (i + 1, j - 1)$. After $l_i - 1$ iterations, we reach $(l_i, 1)$, which is already in the bottom row.
- We are at the position $(l_i, 1)$ with $j = 1$. Since $(1, l_i)$ is in a long row and it will remain in a long row when it reaches the bottom row for the first time, we apply the second rule of Proposition 1 once: $\pi(i, 1) = (T_2\pi)(1, m + 2 - i)$ for partitions with $m + 1$ cells in the bottom row. Substituting $i = l_i$, we obtain the new position $(1, m + 2 - l_i)$.

- From position $(1, m + 2 - l_i)$ with $j > 1$, we apply the first rule ($j > 1$ case) exactly $m + 1 - l_i$ times. Each iteration moves $(i, j) \rightarrow (i + 1, j - 1)$. After $m + 1 - l_i$ steps, we reach $(m + 2 - l_i, 1)$, visiting the bottom row for the second time.

Previous position	Number of iterations of T_2^{-1}	New position
$(1, l_i)$	$l_i - 1$	$(l_i, 1)$
$(l_i, 1)$	1	$(1, m + 2 - l_i)$
$(1, m + 2 - l_i)$	$m + 1 - l_i$	$(m + 2 - l_i, 1)$

Table 5: Position changes for Case 5

The rule in Proposition 1 for partitions with $m + 1$ cells in the bottom row is applied once during the first visit to the bottom row. After a total of $m + 1$ iterations of Proposition 1 (across all steps), the algorithm visits the bottom row for the second time at position $(m + 2 - l_i, 1)$. Thus, $\pi(1, l_i) = (T_2^{m+1}\pi)(m + 2 - l_i, 1)$ and using Definition 3, we have $\sigma_\pi(\pi(m + 2 - l_i, 1)) = (T_2^{m+1}\pi)(m + 2 - l_i, 1) = \pi(1, l_i)$.

For $q = h_i$, we can follow the same steps with starting position, $(1, h_i)$. Replacing l_i with h_i from above result we obtain, $\pi(1, h_i) = (T_2^{m+1}\pi)(m + 2 - h_i, 1)$ and using Definition 3, we have $\sigma_\pi(\pi(m + 2 - h_i, 1)) = (T_2^{m+1}\pi)(m + 2 - h_i, 1) = \pi(1, h_i)$.

■

Having established the equivalence $\sigma_\pi^{col} = \sigma_\pi$, we now turn to the crucial task of determining the order of this permutation. The cycle structure of σ_π governs the long-term behavior of colored partitions under repeated applications of T_2 , with the orbit size being determined by the least common multiple of cycle lengths. To compute $\text{ord}(\sigma_\pi)$, we must analyze the sizes of the colored regions and understand how the propagation mechanism affects cycle lengths. This analysis begins with a fundamental lemma about the initial colored sets in triangular partitions, which provides the foundation for our general formula.

4.4 The cycle length of σ_π^{col}

Lemma 2. *Given $n \geq 1$, write n uniquely as $T_m + m'$ for some integer m' with $0 < m' \leq m$. In the cell coloring procedure for π (Algorithm 1), the sizes of the initially colored sets $X^t(E) \cup X^{m'-1-t}(E)$ for $t = 0, 1, \dots, \lfloor m'/2 \rfloor - 1$ before propagation are as follows:*

1. *If $m' (\neq 0)$ is even, then all colored sets have the same size, $m' + 1$.*
2. *If m' is odd, then the first $\lfloor m'/2 \rfloor$ colored sets have the same size, while the last colored set has exactly half the size of the others.*

Proof. We analyze the sizes of the colored sets $X^t(E) \cup X^{m'-1-t}(E)$ for $t = 0, 1, \dots, \lfloor m'/2 \rfloor - 1$.

First, observe that for any $k \in \{0, 1, \dots, m' - 1\}$, the set $X^k(E)$ contains exactly $m' - k$ cells. This follows from the definition of the X operator, which reduces the number of cells by one with each application.

For a given t , the colored set is $X^t(E) \cup X^{m'-1-t}(E)$. The sizes of these individual sets are:

$$\begin{aligned} |X^t(E)| &= m' - t \\ |X^{m'-1-t}(E)| &= m' - (m' - 1 - t) = t + 1. \end{aligned}$$

The sets $X^t(E)$ and $X^{m'-1-t}(E)$ are disjoint when $t \neq m' - 1 - t$, which occurs when $2t \neq m' - 1$.

Case 1: $m'(\neq 0)$ is even. The number of colors is $m'/2$. For each $t = 0, 1, \dots, m'/2 - 1$, we have $t \neq m' - 1 - t$ since $2t \neq m' - 1$ (as $m' - 1$ is odd). Thus the sets are disjoint and the total size is $(m' - t) + (t + 1) = m' + 1$. All colored sets have equal size $m' + 1$.

Case 2: m' is odd. The number of colors is $(m' + 1)/2$. For $t = 0, 1, \dots, (m' - 3)/2$, we have $t \neq m' - 1 - t$, so the sets are disjoint with size $(m' - t) + (t + 1) = m' + 1$. For $t = (m' - 1)/2$ (the last color), we have $t = m' - 1 - t$, so $X^t(E) = X^{m'-1-t}(E)$ and the union is simply $X^{(m'-1)/2}(E)$ with size $m' - (m' - 1)/2 = (m' + 1)/2$, which is exactly half of $m' + 1$. ■

Example 11. Consider the partition in Figure 11 of $n = 86$ with $m = 12$ and $m' = 8$. The diagonal $\text{diag}[13]$ contains $m' = 8$ extra cells. This illustrates Case 1 of Lemma 2, where $m'(\neq 0)$ is even. According to the lemma, we expect all colored sets to have the same size.

With $m' = 8$ (even), the algorithm uses $8/2 = 4$ colors. Each colored set should have size $m' + 1 = 9$ cells. The coloring pattern shows:

- Yellow (C_1): 9 cells (set $X^0(E) \cup X^7(E)$)
- Green (C_2): 9 cells (set $X^1(E) \cup X^6(E)$)
- Orange (C_3): 9 cells (set $X^2(E) \cup X^5(E)$)
- Cyan (C_4): 9 cells (set $X^3(E) \cup X^4(E)$)

This confirms that all four initially colored intersection points have equal size, as predicted by the lemma. □

Theorem 6. For any positive integer $n \geq 1$, let m be the unique positive integer such that

$$T_m \leq n < T_{m+1},$$

and let $m' = n - T_m$. Assume that if $m' \neq 0$, the partition $\pi \in \mathcal{P}_n^*$ has an extra cell in the first column. Then

$$\text{ord}(\sigma_\pi) = \text{ord}(\sigma_\pi^{\text{col}}) = \begin{cases} 1, & \text{if } m' = 0, \\ m + 1, & \text{if } m' = 1, \\ 2m - m' + 3, & \text{if } 2 \leq m' \leq m. \end{cases}$$

Proof. We consider three cases based on the value of $m' = n - T_m$.

Case 1: $m' = 0$.

When $m' = 0$, π is the full triangular partition of size T_m . In this case, all cells lie in the first m diagonals and are fixed points under T_2^{m+1} by Theorem 5. Therefore, σ_π is the identity permutation and $\text{ord}(\sigma_\pi) = 1$.

Case 2: $m' = 1$.

When $m' = 1$, there is exactly one extra cell in diagonal $m + 1$. Assume this cell is at position $(1, m + 1)$, so $h_1 = m + 1$ and $l_1 = m + 1$. The coloring algorithm produces one non-trivial cycle containing all cells in the first column, which has length $m + 1$. All other cells are fixed points. Thus $\text{ord}(\sigma_\pi^{\text{col}}) = m + 1$.

Case 3: $2 \leq m' \leq m$.

Assume there is an extra cell at $(1, m + 1)$, so that $h_1 = m + 1$. Let $h_1 > h_2 > \dots > h_{m'}$ be the heights of the rows with extra cells.

We apply the cell coloring procedure (Algorithm 1) to partition the cells of π into $r = \lceil m'/2 \rceil$ color classes $C_1, \dots, C_r$ and uncolored cells. Note that even though the algorithm considers t values from 0 to $m' - 1$, it only uses $\lceil m'/2 \rceil$ colors by pairing sets $X^t(E)$ and $X^{m'-1-t}(E)$ for $t = 0, \dots, \lceil m'/2 \rceil - 1$.

Initial Coloring. By Lemma 2, the initial sizes of the colored sets before propagation are:

- If $m'(\neq 0)$ is even: all r color classes have initial size $m' + 1$.
- If m' is odd: color classes $C_1, \dots, C_{r-1}$ have initial size $m' + 1$, and C_r has initial size $(m' + 1)/2$.

Propagation Analysis. Since long rows follow $h_1 = m + 1 > h_2 > \dots > h_{m'}$ and the long columns follow $m + 2 - h_1 < m + 2 - h_2 < \dots < m + 2 - h_{m'}$, then $m + 2 - h_1 = 1$ and $m + 2 - h_{m'}$ is the rightmost long column.

Let R_0 be the region covering of the cells that are strictly below long row $h_{m'}$ and strictly to the right of column $m + 2 - h_{m'}$. Let $k_0 = h_{m'} - 1$ be the number of short rows below long row $h_{m'}$. The initially colored cells from Algorithm 1 are distributed such that:

- When m' is even, each of the $r = m'/2$ colors appears exactly twice.
- When m' is odd, colors C_1 through C_{r-1} appear twice, and C_r appears once.

Each colored cell propagates downward through all k_0 short rows. Thus, downward propagation adds:

- Colors C_1 through C_{r-1} : $2k_0$ cells each.
- Color C_r : $2k_0$ if m' is even and k_0 if m' is odd.

Note that there is no leftward propagation in region R_0 and all cells to the right of Column $m + 2 - h_{m'}$ are uncolored.

Let the region R_t for $t = 1, \dots, m' - 1$, consist of the cells that are

- Strictly between row $h_{m'-t}$ and row $h_{m'-t+1}$,
- Strictly between column $m + 2 - h_{m'-t}$ and column $m + 2 - h_{m'-t+1}$.

Note that the region R_t contains $k_t = h_{m'-t-1} - h_{m'-t} - 1$ short rows and short columns. The total number of short rows is $\sum_{t=0}^{m'-1} k_t = m + 1 - m'$.

Downward Propagation in R_t . By symmetry, in row $h_{m'-t}$, the initially colored cells, $X^k(E) \cup X^{m'-1-k}(E)$ for $k = 0, 1, \dots, \lceil m'/2 \rceil - 1$ are distributed as follows:

- Colors C_1 through C_t : Appear once each.
- Colors C_{t+1} through C_{r-1} : Appear twice each.
- Color C_r : Appears twice if m' is even and once if m' is odd.

Each colored cell propagates downward through k_t short rows, giving:

- Colors C_1 through C_t : k_t cells each.
- Colors C_{t+1} through C_{r-1} : $2k_t$ cells each.
- Color C_r : $2k_t$ if m' is even and k_t if m' is odd.

Leftward Propagation in R_t . By symmetry, in column $m + 2 - h_{m'-t}$, the initially colored cells, $X^k(E) \cup X^{m'-1-k}(E)$ for $k = 0, 1, \dots, \lceil m'/2 \rceil - 1$ are distributed as colors C_1 through C_t appear once each (total t cells). Each such cell propagates leftward through k_t short columns, adding exactly k_t cells to each of colors C_1 through C_t . For colors C_{t+1} through C_r , there are no colored cells in the column $m + 2 - h_{m'-t}$ to generate leftward propagation in this region.

Total Propagation in Region R_t for $1 \leq t \leq m' - 1$. Combining downward and leftward propagation:

- Colors C_1 through C_t : k_t (downward) + k_t (leftward) = $2k_t$.
- Colors C_{t+1} through C_{r-1} : $2k_t$ (downward) + 0 (leftward) = $2k_t$.
- Color C_r : $2k_t$ if m' is even and k_t if m' is odd.

Summing over all regions $t = 0$ to $m' - 1$, and noting that $\sum_{t=0}^{m'-1} k_t = m + 1 - m'$, we obtain the total propagation gain:

- For colors C_1 through C_{r-1} : $2(m + 1 - m')$.
- For color C_r : $\begin{cases} 2(m + 1 - m') & \text{if } m' \text{ even} \\ m + 1 - m' & \text{if } m' \text{ odd.} \end{cases}$

Therefore, the total number of cells in each color class is:

- If m' is even:

$$|C_i| = (m' + 1) + 2(m + 1 - m') = 2m - m' + 3 \quad \text{for } i = 1, \dots, r.$$

- If m' is odd:

$$|C_i| = (m' + 1) + 2(m + 1 - m') = 2m - m' + 3 \quad \text{for } i = 1, \dots, r - 1,$$

and

$$|C_r| = \frac{m' + 1}{2} + (m + 1 - m') = \frac{2m - m' + 3}{2}.$$

The order of σ_π^{col} . By the cycle tracing algorithm and Theorem 5, the permutation σ_π acts on the colored cells as a product of disjoint cycles, with each color class forming exactly one cycle under σ_π . Therefore, the cycle lengths of σ_π are exactly the sizes of the color classes. The order of σ_π is the least common multiple of these cycle lengths:

- If m' is even: all cycles have length $2m - m' + 3$, so

$$\text{ord}(\sigma_\pi^{\text{col}}) = 2m - m' + 3.$$

- If m' is odd: we have $r - 1$ cycles of length $2m - m' + 3$ and one cycle of length $(2m - m' + 3)/2$. Since m' is odd and $2 \leq m' \leq m$, the number $2m - m' + 3$ is even. Therefore, $(2m - m' + 3)/2$ divides $2m - m' + 3$, and

$$\text{ord}(\sigma_\pi^{\text{col}}) = \text{lcm}(2m - m' + 3, (2m - m' + 3)/2) = 2m - m' + 3.$$

This completes the proof for all cases. ■

Example 12 (Explanation of the proof of Theorem 6 using Figure 11). Building upon Example 8, we now demonstrate how the proof of Theorem 6 applies to the colored partition in Figure 11 with $n = 86$, $m = 12$, and $m' = 8$.

Since $m' = 8$ is even, we need $8/2 = 4$ colors: Yellow (C_1), Green (C_2), Orange (C_3), Cyan (C_4) and since $2 \leq m' \leq m$, we apply Case 3 of the theorem 6:

$$\text{ord}(\sigma_\pi) = 2m - m' + 3 = 2 \times 12 - 8 + 3 = 24 - 8 + 3 = 19.$$

From the set $E = \{(1, 13), (2, 12), (3, 11), (4, 10), (5, 9), (6, 8), (9, 5), (12, 2)\}$, we extract the h_k values,

- | | |
|--|---|
| • $h_8 = 2$ (lowest row with extra cell) | • $h_4 = 10$ |
| • $h_7 = 5$ | • $h_3 = 11$ |
| • $h_6 = 8$ | • $h_2 = 12$ |
| • $h_5 = 9$ | • $h_1 = 13$ (highest row with extra cell). |

Regions and Propagation Analysis. The regions between consecutive long rows and long columns (from bottom to top) are:

- **Region R_0 :** Strictly below $h_8 = 2$ (lowest long row) and strictly to the right of column $m + 2 - h_8 = 12$.
Contains $k_0 = 1$ short row (row 1) and no short columns. All colors $C_1, \dots, C_4$ propagate $2k_0 = 2$ cells downwards.
- **Region R_1 :** Strictly between row $h_7 = 5$ and row $h_8 = 2$.
Strictly between column $m + 2 - h_8 = 12$ and column $m + 2 - h_7 = 9$.
Contains $k_1 = 2$ short rows (rows 3 and 4) and short columns (columns 10 and 11).
 - Color C_1 : k_1 (downward) + k_1 (leftward) = $2k_1 = 4$ cells.
 - Colors C_2 through C_4 : $2k_1$ (downward) + 0 (leftward) = $2k_1 = 4$ cells.
- **Region R_2 :** Strictly between row $h_6 = 8$ and row $h_7 = 5$.
Strictly between column $m + 2 - h_6 = 6$ and column $m + 2 - h_5 = 9$.
Contains $k_2 = 2$ short rows (rows 6 and 7) and short columns (columns 7 and 8).
 - Color C_1 and C_2 : k_2 (downward) + k_2 (leftward) = $2k_2 = 4$ cells.
 - Colors C_3 and C_4 : $2k_2 = 4$ (downward) + 0 (leftward) = $2k_2 = 4$ cells.

- **Region R_t for $t = 3, \dots, 7$:** Strictly between row $h_{m'-t}$ and row $h_{m'-t+1}$.
Strictly between column $m + 2 - h_{m'-t}$ and column $m + 2 - h_{m'-t+1}$.
Contains $k_t = 0$ short rows and short columns.
Colors C_1 through C_t : $k_t(\text{downward}) + k_t(\text{leftward}) = 2k_t = 0$ cells.
Colors C_{t+1} through C_7 : $2k_t(\text{downward}) + 0(\text{leftward}) = 2k_t = 0$ cells..

Note that even though the algorithm considers t values from 0 to 7, it only uses $\lceil m'/2 \rceil = 4$ colors by pairing sets $X^t(E)$ and $X^{7-t}(E)$ for $t = 0, \dots, 7$.

The total number of short rows is

$$\sum_{t=0}^7 k_t = 1 + 2 + 2 + 0 + 0 + 0 + 0 = 5,$$

which equals $m + 1 - m' = 13 - 8 = 5$, as expected.

Initial Coloring. As established in Lemma 2, the initial colored sets before propagation have:

- $r = m'/2 = 4$ color classes (Yellow, Green, Orange, Cyan)
- Each color class has initial size $m' + 1 = 9$ cells.

Propagation Analysis. The propagation adds $2(m + 1 - m') = 2(13 - 8) = 10$ cells to each color class, giving final sizes

$$|C_i| = 9 + 10 = 19 \quad \text{for each color class } i = 1, \dots, 4.$$

Cycle Structure and Order. By the cycle tracing algorithm and Theorem 5, each color class forms exactly one cycle under σ_π . Since all four cycles have length 19, the order of σ_π is

$$\text{ord}(\sigma_\pi) = \text{lcm}(19, 19, 19, 19) = 19.$$

□

Corollary 2. Let $\pi \in \mathcal{P}_n^*$ be a periodic point with $n = T_m + m'$ and $0 \leq m' \leq m$. Then the orbit size of π under T_2 divides $(m + 1) \cdot \text{ord}(\sigma_\pi)$, where

$$\text{ord}(\sigma_\pi) = \begin{cases} 1, & \text{if } m' = 0, \\ m + 1, & \text{if } m' = 1, \\ 2m - m' + 3, & \text{if } 2 \leq m' \leq m. \end{cases}$$

Proof. For $m' = 0$, the result is trivial since $\text{ord}(\sigma_\pi) = 1$ and the orbit size divides $m + 1$.

For $m' > 0$, without loss of generality, we can pick a partition from the orbit of the periodic partition $\pi \in \mathcal{P}_n^*$, that has an extra cell in the first column, ensuring that the cell $(1, m + 1)$ is present and lies on the $(m + 1)$ -th diagonal. Then the corollary is a direct consequence of Theorem 4 and Theorem 6. Theorem 4 establishes that the orbit size of π under T_2 divides $(m + 1) \cdot \text{ord}(\sigma_\pi)$, and Theorem 6 provides the explicit value of $\text{ord}(\sigma_\pi)$ in terms of m and m' . ■

Corollary 2 shows that the size $|O_{T_2}(\pi)|$ is a divisor of $(m + 1) \cdot \text{ord}(\sigma_\pi)$. We computed the size of $|O_{T_2}(\pi)|$ for all periodic partitions $\pi \in \mathcal{P}_n^*$, up to $n = 100$. Based on computational evidence, we now state our main conjecture regarding the orbit size of π under T_2 .

Conjecture 1 (Orbit size under T_2). Let $\pi \in \mathcal{P}_n^*$ be a periodic point under the action T_2 , with $n \geq 1$ and $n = T_m + m'$ where $0 \leq m' \leq m$ and $T_m = \frac{m(m+1)}{2}$. Let ℓ be the period of the underlying unlabeled partition under T , and let $r = (m+1)/\ell$. Then, the orbit size of π under T_2 is

$$|O_{T_2}(\pi)| = \ell \cdot \text{lcm}(\text{ord}(\sigma_\pi), \delta),$$

where

$$\delta = \begin{cases} r, & \text{if } m - m' \geq 2 \\ 1, & \text{otherwise.} \end{cases}$$

Here, $\text{ord}(\sigma_\pi)$ is as defined in Definition 3 and is given explicitly by Theorem 6.

We finish by giving several examples that illustrate the variance in orbit sizes in practice.

Example 13 ($n = 8$). For $n = 8$, we have $T_3 = 6 \leq 8 < T_4 = 10$, so $m = 3$ and $m' = 2$. From Theorem 6, since $2 \leq m' \leq m$, we compute

$$\text{ord}(\sigma_\pi) = 2m - m' + 3 = 2 \cdot 3 - 2 + 3 = 7.$$

The possible periods ℓ are determined by necklaces with $m' = 2$ black beads and $m + 1 - m' = 2$ white beads. The necklace analysis shows $\ell \in \{2, 4\}$. Since $m - m' = 1 < 2$, we have $\delta = 1$. We have two cases:

- **Case** $\ell = 2$: Then $r = (m+1)/\ell = 4/2 = 2$. Thus

$$|O_{T_2}(\pi)| = \ell \cdot \text{lcm}(\text{ord}(\sigma_\pi), \delta) = 2 \cdot \text{lcm}(7, 1) = 14.$$

- **Case** $\ell = 4$: Then $r = 4/4 = 1$. Thus

$$|O_{T_2}(\pi)| = 4 \cdot \text{lcm}(7, 1) = 28.$$

Thus for $n = 8$, the possible orbit sizes are 14 and 28. □

Example 14 ($n = 18$). For $n = 18$, we have $T_5 = 15 \leq 18 < T_6 = 21$, so $m = 5$ and $m' = 3$. From Theorem 6:

$$\text{ord}(\sigma_\pi) = 2m - m' + 3 = 2 \cdot 5 - 3 + 3 = 10.$$

The possible periods ℓ are determined by necklaces with $m' = 3$ black beads and $m + 1 - m' = 3$ white beads. The necklace analysis shows $\ell \in \{2, 6\}$. Since $m - m' = 2 \geq 2$, we have $\delta = r$. We have two cases:

- **Case** $\ell = 2$: Then $r = 6/2 = 3$, so $\delta = 3$. Thus

$$|O_{T_2}(\pi)| = 2 \cdot \text{lcm}(10, 3) = 60.$$

- **Case** $\ell = 6$: Then $r = 6/6 = 1$, so $\delta = 1$. Thus

$$|O_{T_2}(\pi)| = 6 \cdot \text{lcm}(10, 1) = 60.$$

Thus for $n = 18$, the orbit size is always 60. □

Example 15 ($n = 32$). For $n = 32$, we have $T_7 = 28 \leq 32 < T_8 = 36$, so $m = 7$ and $m' = 4$. From Theorem 6:

$$\text{ord}(\sigma_\pi) = 2m - m' + 3 = 2 \cdot 7 - 4 + 3 = 13.$$

The possible periods ℓ are determined by necklaces with $m' = 4$ black beads and $m + 1 - m' = 4$ white beads. The necklace analysis shows $\ell \in \{2, 4, 8\}$. Since $m - m' = 3 \geq 2$, we have $\delta = r$. We have three cases:

- **Case** $\ell = 2$: Then $r = 8/2 = 4$, so $\delta = 4$. Thus

$$|O_{T_2}(\pi)| = 2 \cdot \text{lcm}(13, 4) = 104.$$

- **Case** $\ell = 4$: Then $r = 8/4 = 2$, so $\delta = 2$. Thus

$$|O_{T_2}(\pi)| = 4 \cdot \text{lcm}(13, 2) = 104.$$

- **Case** $\ell = 8$: Then $r = 8/8 = 1$, so $\delta = 1$. Thus

$$|O_{T_2}(\pi)| = 8 \cdot \text{lcm}(13, 1) = 104.$$

Thus for $n = 32$, all orbit sizes are 104, regardless of the period ℓ . □

Example 16 ($n = 28$). For $n = 28$, we have $T_7 = 28 \leq 28 < T_8 = 36$, so $m = 7$ and $m' = 0$. From Theorem 6, since $m' = 0$, we compute

$$\text{ord}(\sigma_\pi) = 1.$$

The underlying unlabeled partition is the triangular partition $(7, 6, 5, 4, 3, 2, 1)$, which is a fixed point under T , so $\ell = 1$. Then $r = (m + 1)/\ell = 8/1 = 8$. Since $m - m' = 7 \geq 2$, we have $\delta = r$. So,

$$|O_{T_2}(\pi)| = 1 \cdot \text{lcm}(1, 8) = 8.$$

Thus for $n = 28$, the orbit size under T_2 is 8. □

Example 17 ($n = 14$: Case $m' = m$ with no uncolored cells). For $n = 14$, we have $T_4 = 10 \leq 14 < T_5 = 15$, so $m = 4$ and $m' = 4$. From Theorem 6, since $2 \leq m' \leq m$, we compute

$$\text{ord}(\sigma_\pi) = 2m - m' + 3 = 2 \cdot 4 - 4 + 3 = 7.$$

The underlying unlabeled partition has period ℓ determined by necklace analysis with $m' = 4$ black beads and $m + 1 - m' = 1$ white bead. Since $\gcd(m + 1, m') = \gcd(5, 4) = 1$, there is exactly one necklace with period $\ell = m + 1 = 5$. Thus $\ell = 5$ and $r = (m + 1)/\ell = 5/5 = 1$. Since $m - m' = 0 < 2$, we have $\delta = 1$. So,

$$|O_{T_2}(\pi)| = 5 \cdot \text{lcm}(7, 1) = 35.$$

Thus for $n = 14$ with $m = 4$ and $m' = 4$, the orbit size under T_2 is 35. □

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