

Bulgarian solitaire with distinguishable cells

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Abstract

Bulgarian solitaire is a combinatorial game on integer partitions whose long-term behavior is well understood. We introduce four natural generalizations acting on partitions with distinguishable cells. These actions differ in whether the top or bottom cell is removed from each column and in the order in which the removed cells are arranged to form a new column. We show that a configuration is periodic under any of these actions if and only if its underlying partition is periodic under classical Bulgarian solitaire, and we determine the exact sizes of the periodic orbits for all four actions. A key ingredient is a coloring algorithm on Ferrers diagrams that encodes the cycle structure of an induced permutation of the labels.

1 Introduction

Bulgarian solitaire is a mathematical game played with cards. One starts with n cards divided into stacks and placed face down. One step of the game consists of taking one card from every stack to form a new stack. The game is won when the sequence of arrangements becomes periodic. Figure 1 shows one move in a game with 5 cards, from the configuration with stack sizes $(3, 1, 1)$ to the configuration $(3, 2)$. Because the cards are indistinguishable, the configuration is encoded only by the stack sizes. These stack sizes are traditionally ordered in weakly decreasing order and represented as a partition of n . Thus, Bulgarian solitaire defines a discrete dynamical system on the set $\mathcal{P}_n$ of partitions of n .

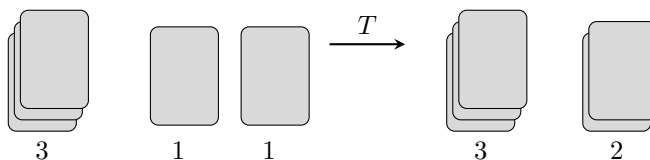


Figure 1: One move of Bulgarian solitaire from $(3, 1, 1)$ to $(3, 2)$. Remove one card from each stack to create a stack of size 3; the original stack of size 3 becomes a stack of size 2, and the two singleton stacks disappear.

The game was introduced by Karatsuba during a visit to the Bulgarian Academy of Sciences in May 1980 [7]. The following year, Toom provided the first proof, demonstrating that for certain numbers of cards, the game inevitably settles into a “triangular configuration” [28]. Bojanov and Eriksson published treatments that same year [3, 8]. Gardner popularized the game in his *Scientific American* column, bringing it to an international audience [14]. Akin and Davis discussed the game in *The American Mathematical Monthly* [1], and Hopkins reviewed its 30 years of history in [19].

Subsequent work examined the cycles and dynamics of Bulgarian solitaire. Brandt studied the cycles formed by the stacks in Bulgarian solitaire and found the number of distinct cycles (orbits) for partitions

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of n [5]. Later work explored cycle counts and periodicity [15, 17, 21]. Other studies analyzed game-tree level sizes and depth generating functions [10, 16], as well as the limiting behavior of growing orbits [26]. Related work on partitions considers column-to-row operations [18], an inductive proof of a result concerning Bulgarian solitaire [23], and connections of the game to Young tableaux [13] and to Owari marching groups [4].

In addition to the studies of the classical game, there are many variants of Bulgarian solitaire, including Austrian and Montreal solitaire, multiplayer forms and proposed variants [2, 6, 9, 20, 25, 29]. More recently, Meyerowitz et al. introduced Floridian solitaire [22]. Probabilistic studies examine random Bulgarian solitaire, its random q -proportion variant, and the limit shape of stochastic Bulgarian solitaire [11, 12, 27]. The surveys [7, 24] provide broader reviews of the mathematics of the game and its variants.

In the classical game, the cards are indistinguishable and only the stack sizes are tracked. This paper asks what happens to individual cards as the game proceeds. For this purpose, we extend the Bulgarian solitaire to partitions with distinguishable cells, denoted by $\mathcal{P}_n^*$, and define four natural actions T_1, T_2, T_3, T_4 . The actions differ in whether cards are removed from the top or the bottom of the stack and in the order in which the removed cards are ordered to form a new stack. Our main results characterize the orbit sizes of these actions. In particular, we show that a configuration is periodic under any of the four actions if and only if its underlying partition is periodic under classical Bulgarian solitaire, and we determine the exact sizes of the resulting orbits.

The paper is organized as follows. Section 2 reviews the necessary definitions of partitions, Ferrers diagrams, and classical Bulgarian solitaire, including the characterization of periodic points. In Section 3, we introduce the four natural actions on partitions with distinguishable cells, and we show how they are related to the classical Bulgarian solitaire and to one another through reversal. This reduces our work to analyzing the orbits of T_1 and T_2 only. In Section 4, we begin the orbit size analysis: we determine the orbit sizes for T_1 and we develop a coloring algorithm to analyze T_2 . In Section 5, we use that analysis to determine the exact orbit sizes under T_2 . Finally, in Section 6, we show how conjugation interacts with our actions and how it reverses the direction of periodic orbits.

2 Background

In this section, we recall the necessary definitions of partitions, Ferrers diagrams, and classical Bulgarian solitaire, in particular the characterization of periodic points, which will be used to analyze the generalization introduced later.

2.1 Partitions and Ferrers diagrams

Definition 2.1. A partition of a positive integer n is a finite sequence $\pi = (\pi_1, \pi_2, \dots, \pi_k)$ of positive integers such that $\pi_1 \geq \pi_2 \geq \dots \geq \pi_k$ and $\sum_{i=1}^k \pi_i = n$. The integers π_i are called the parts of the partition, and $k = \ell(\pi)$ is called the length of the partition. The set of all partitions of n is denoted by $\mathcal{P}_n$.

A partition $\pi = (\pi_1, \pi_2, \dots, \pi_k)$ of n is traditionally represented by its Ferrers diagram: a left-aligned array of cells in which the i -th row from the top contains π_i cells. In our pictorial representation, we rotate this diagram by 90° counterclockwise so that the parts correspond to columns (representing stacks of cards) rather than rows (e.g., Figure 2). We refer to the cells of this diagram using Euclidean coordinates, with the bottom-left cell at position $(1, 1)$.

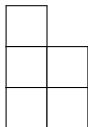


Figure 2: Rotated Ferrers diagram of $\pi = (3, 2) \in \mathcal{P}_5$.

For two partitions μ and π , we define their union $\mu \cup \pi$ to be the partition obtained by combining the parts of μ with those of π and arranging the resulting entries in weakly decreasing order. For example, $(3, 2) \cup (5, 4, 2, 1) = (5, 4, 3, 2, 2, 1)$.

2.2 Classical Bulgarian solitaire

Bulgarian solitaire is a discrete dynamical system defined on $\mathcal{P}_n$. The system evolves through an iterative process governed by the following rule: from a given partition, remove one cell from each part (column) and use the removed cells to form a new part. Formally, Bulgarian solitaire is the map $T : \mathcal{P}_n \rightarrow \mathcal{P}_n$ defined by

$$T(\pi) = (\pi_1 - 1, \pi_2 - 1, \dots, \pi_{\ell(\pi)} - 1) \cup (\ell(\pi)),$$

where parts of size zero are omitted.

The set

$$O_T(\pi) = \{\pi, T(\pi), T^2(\pi), \dots\}$$

is called the orbit of π under T . Since $\mathcal{P}_n$ is finite, the sequence of iterates is eventually periodic.

A partition π is said to be *periodic* under the transformation T if there exists a positive integer ℓ such that

$$T^\ell(\pi) = \pi.$$

The smallest such integer ℓ is called the *period* of π under T .

Definition 2.2. For a fixed positive integer k , the k -th upper diagonal, denoted by $\text{diag}[k]$, is the set of positions

$$\text{diag}[k] = \{(1, k), (2, k-1), \dots, (k, 1)\}.$$

For a positive integer n , let m be the unique integer such that the m -th triangular number $\Delta_m = \frac{m(m+1)}{2}$ satisfies

$$\Delta_m \leq n < \Delta_{m+1}.$$

Define $m' = n - \Delta_m$.

The periodic points of T are characterized as follows.

Theorem 2.3 ([17]). Let $n \geq 1$, let m be the unique integer such that $\Delta_m \leq n < \Delta_{m+1}$ and let $m' = n - \Delta_m$. Let π be a partition of n , and let F denote its Ferrers diagram. Then π is a periodic point of T if and only if

$$|F \cap \text{diag}[k]| = \begin{cases} k, & 1 \leq k \leq m, \\ m', & k = m+1, \\ 0, & k > m+1. \end{cases}$$

In the special case where $n = \Delta_m$, $\mathcal{P}_n$ contains only one periodic point, namely,

$$(m, m-1, m-2, \dots, 2, 1).$$

When n is not triangular, the period ℓ of a periodic partition divides $m+1$. As can be gleaned from Example 2.5, for periodic points, the newly formed part is always at least as large as all remaining parts and can be placed in the first position [5]. The overall change in the Ferrers shape amounts to a cyclic downward shift of the cells in $\text{diag}[m+1]$.

The number of distinct periodic orbits of $\mathcal{P}_n$ under the action of T is equal to the number of necklaces consisting of m' black beads and $m+1-m'$ white beads, where the symmetry group is the cyclic group of order $m+1$. Their number is given by the following elegant formula.

Theorem 2.4 ([5]). Given $n \geq 1$, write n uniquely as $\Delta_m + m'$ for some integer m' with $0 \leq m' \leq m$. The number of distinct periodic orbits in $\mathcal{P}_n$ under T is

$$\frac{1}{m+1} \sum_{d|\gcd(m+1, m')} \phi(d) \left(\frac{\frac{m+1}{d}}{\frac{m'}{d}} \right),$$

where ϕ is Euler's totient function.

Example 2.5. Figure 3 illustrates the orbit of $\pi = (3, 1, 1)$ under T . By Theorem 2.3, π is a periodic point of T . The action of T corresponds to cyclically shifting the cells in $\text{diag}[m+1]$ down the diagonal.

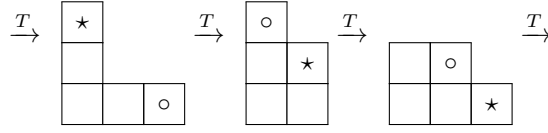


Figure 3: Orbit of $\pi = (3, 1, 1)$ under T .

In this case, $n = 5 = \Delta_2 + 2$, so $m = 2$ and $m' = 2$. The period is $\ell = 3$, which divides $m + 1$. This orbit corresponds to the unique necklace with $m' = 2$ black beads and $m + 1 - m' = 1$ white bead, shown in Figure 4.

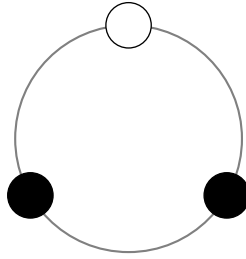


Figure 4: A necklace with 2 black beads and 1 white bead.

The *depth* $D(n)$ of classical Bulgarian solitaire is the maximum, over $\pi \in \mathcal{P}_n$, of the minimum number of applications of T needed to reach a periodic partition. Griggs and Ho [15, Theorems 3.7 and 4.4] proved that $D(\Delta_m) = m(m-1)$ for $m \geq 1$ and $D(\Delta_m + m') \leq m^2 - 2$ for $m \geq 3$ and $1 \leq m' \leq m$, with equality in the latter bound when $m' = m$. For results on counting partitions according to the number of moves required to reach a given periodic orbit, see [16, 26].

3 Four variants of Bulgarian solitaire on $\mathcal{P}_n^\star$

When the cards are turned face up, each card becomes individually identifiable, so each stack is represented by an ordered sequence of distinct labels. This motivates the study of *partitions with distinguishable cells*, where the identity of each card is taken into account.

A partition with distinguishable cells is a Ferrers diagram whose cells are bijectively labeled by a fixed set of n labels. No order conditions are imposed on the labels. We write each part as $\pi_i = (\pi_{i1}, \pi_{i2}, \dots, \pi_{in_i})$, where n_i is the length of the i -th column and the entries represent distinct cells. We depict each π_i as a column in the Ferrers diagram, and π_{ij} denotes the labeled cell in position (i, j) , that is, the cell in the i -th column and j -th row.

We denote the set of all partitions of n with distinguishable cells by $\mathcal{P}_n^\star$. The set of all labeled cells of a partition's Ferrers diagram is denoted by $\mathcal{L}_\pi$.

Example 3.1. Consider the partition with distinguishable cells $\pi^\star = ((a, b, c), (d), (e)) \in \mathcal{P}_5^\star$. It corresponds to the Ferrers diagram shown in Figure 5.

c		
b		
a	d	e

Figure 5: Ferrers diagram of $\pi^* = ((a, b, c), (d), (e))$.

Moreover, we have $\mathcal{L}_{\pi^*} = \{a, b, c, d, e\}$.

For two partitions with distinguishable cells μ^* and π^* with disjoint label sets, define their union $\mu^* \cup \pi^*$ by combining their columns and arranging them in weakly decreasing order of length. The order of the cells within each column is preserved. Among columns of the same length, the original left-to-right order within each partition is preserved, and all columns from π^* are placed before those from μ^* . For example,

$$((a, b, c), (d), (e)) \cup ((A, B, C), (D, E), (F)) = ((A, B, C), (a, b, c), (D, E), (F), (d), (e)).$$

We now introduce four actions, T_i for $i = 1, 2, 3, 4$, acting on $\mathcal{P}_n^*$. Each action is defined by choosing a rule for removing one cell from each column, followed by a rule for arranging the removed cells into a new column. There are two natural ways to remove cells: either extract the bottom cell π_{i1} from each column or the top cell π_{in_i} . Likewise, there are two natural methods for placing these cells: stack them in the new column from bottom to top, following their original left-to-right order, or arrange them from top to bottom. The four actions arise from all possible combinations of these removal and placement choices, and their formal definitions are given below.

Definition 3.2. Let $\pi \in \mathcal{P}_n^*$ be

$$\pi = ((\pi_{11}, \pi_{12}, \dots, \pi_{1n_1}), (\pi_{21}, \pi_{22}, \dots, \pi_{2n_2}), \dots, (\pi_{k1}, \pi_{k2}, \dots, \pi_{kn_k})).$$

The actions $T_i : \mathcal{P}_n^* \rightarrow \mathcal{P}_n^*$, for $i = 1, 2, 3, 4$, are defined by

$$\begin{aligned} T_1(\pi) &= ((\pi_{12}, \dots, \pi_{1n_1}), (\pi_{22}, \dots, \pi_{2n_2}), \dots, (\pi_{k2}, \dots, \pi_{kn_k})) \cup (\pi_{11}, \pi_{21}, \dots, \pi_{k1}), \\ T_2(\pi) &= ((\pi_{12}, \dots, \pi_{1n_1}), (\pi_{22}, \dots, \pi_{2n_2}), \dots, (\pi_{k2}, \dots, \pi_{kn_k})) \cup (\pi_{k1}, \pi_{k-1,1}, \dots, \pi_{11}), \\ T_3(\pi) &= ((\pi_{11}, \dots, \pi_{1,n_1-1}), (\pi_{21}, \dots, \pi_{2,n_2-1}), \dots, (\pi_{k1}, \dots, \pi_{k,n_k-1})) \cup (\pi_{1n_1}, \pi_{2n_2}, \dots, \pi_{kn_k}), \\ T_4(\pi) &= ((\pi_{11}, \dots, \pi_{1,n_1-1}), (\pi_{21}, \dots, \pi_{2,n_2-1}), \dots, (\pi_{k1}, \dots, \pi_{k,n_k-1})) \cup (\pi_{kn_k}, \pi_{k-1,n_k-1}, \dots, \pi_{1n_1}). \end{aligned}$$

Empty columns are omitted. In these formulas, the tuple to the right of $\cup$ is regarded as a one-column partition.

In other words, T_1 and T_2 remove the bottom cell π_{i1} from each column. They differ in how the removed cells are stacked: T_1 preserves the left-to-right order, while T_2 reverses it. Similarly, T_3 and T_4 remove the top cell π_{in_i} from each column; T_3 stacks them preserving the left-to-right order, while T_4 reverses it.

As proved in Proposition 3.4, a partition with distinguishable cells is periodic under T_i if and only if its underlying partition is periodic under T . As in the classical Bulgarian solitaire case, for periodic partitions the new column is always at least as long as the remaining columns, and hence the union places it as the first column after applying each T_i . Figures 6 and 7 illustrate the orbits of $\pi = ((a, b, c), (d), (e))$ under T_1 and T_2 , respectively. The other two actions are related to these by the reversal symmetry established in Proposition 3.7.

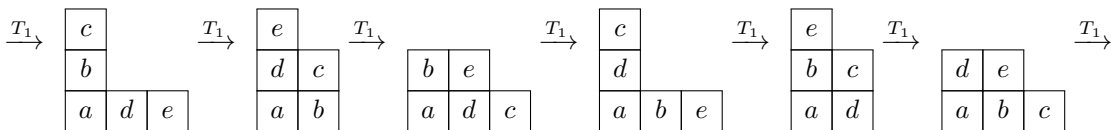


Figure 6: One orbit under the action T_1 .

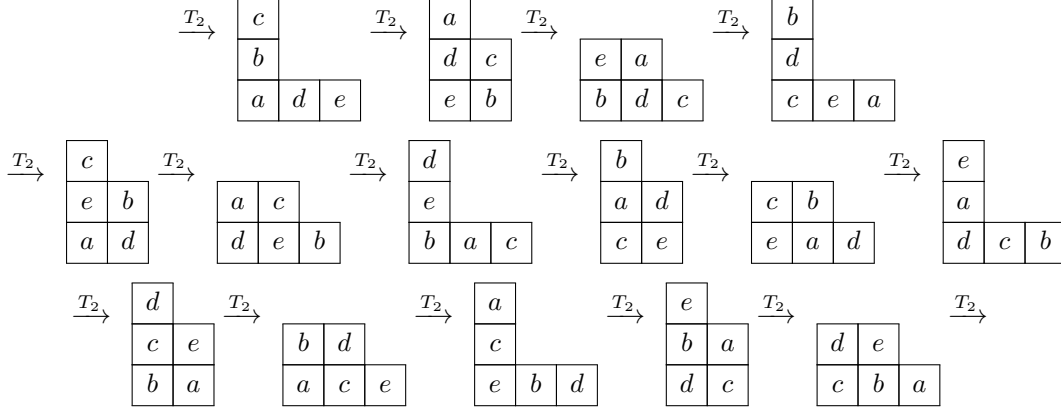


Figure 7: One orbit under the action T_2 .

3.1 Connection to classical Bulgarian solitaire

Each of the four actions T_1, T_2, T_3, T_4 removes one cell from every column and forms a new column of size equal to the number of columns. Consequently, although the labels are permuted differently, the multiset of column sizes evolves exactly as in the classical game. To make this precise, define the *shape* map

$$S : \mathcal{P}_n^* \longrightarrow \mathcal{P}_n, \quad S(\pi^*) = \text{nonincreasing list of column lengths of } \pi^* \in \mathcal{P}_n^*.$$

Proposition 3.3. *For every $\pi^* \in \mathcal{P}_n^*$ and every $i \in \{1, 2, 3, 4\}$,*

$$S(T_i(\pi^*)) = T(S(\pi^*)),$$

where $T : \mathcal{P}_n \rightarrow \mathcal{P}_n$ is the classical Bulgarian solitaire map. In other words, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P}_n^* & \xrightarrow{T_i} & \mathcal{P}_n^* \\ S \downarrow & & \downarrow S \\ \mathcal{P}_n & \xrightarrow{T} & \mathcal{P}_n \end{array}$$

Proof. Let $S(\pi^*) = (a_1, \dots, a_k)$. Each T_i removes one cell from every column and forms a new column from the k removed cells. Thus

$$S(T_i(\pi^*)) = (a_1 - 1, \dots, a_k - 1) \cup (k) = T(S(\pi^*)),$$

where zero parts are omitted and the remaining parts are arranged in weakly decreasing order. $\square$

Proposition 3.4. *For each $i \in \{1, 2, 3, 4\}$, a partition with distinguishable cells π^* is periodic under T_i if and only if its shape $S(\pi^*)$ is periodic under T .*

Proof. $(\Rightarrow)$ First suppose that π^* is periodic under T_i . Then, for some $\ell > 0$,

$$T_i^\ell(\pi^*) = \pi^*.$$

Using Proposition 3.3, namely $S \circ T_i = T \circ S$, we get

$$T^\ell(S(\pi^*)) = S(T_i^\ell(\pi^*)) = S(\pi^*).$$

Thus $S(\pi^*)$ is periodic under T .

$(\Leftarrow)$ Conversely, suppose that $\lambda = S(\pi^*)$ is periodic under T . Then there exists $p > 0$ such that $T^p(\lambda) = \lambda$. For any partition $\nu \in \mathcal{P}_n$, denote the fiber over ν by

$$F_\nu = \{\mu^* \in \mathcal{P}_n^* \mid S(\mu^*) = \nu\}.$$

Each F_ν is finite; indeed $|F_\nu| = n!$ because it consists of all bijections from a set of n distinguishable labels to the cells of the Ferrers diagram of ν .

By Proposition 3.3, we have $T_i(F_\nu) \subseteq F_{T(\nu)}$. Moreover, the map

$$T_i : F_\nu \longrightarrow F_{T(\nu)}$$

is injective: the source shape ν is fixed, so the newly created column has known length $\ell(\nu)$. By the tie-breaking rule, it is the leftmost column of that length. Removing it leaves the nonempty remnants of the original columns in their original order. Reading its labels in the order prescribed by T_i and reinserting them reconstructs the original configuration, including any singleton columns that disappeared. Since $|F_\nu| = |F_{T(\nu)}| = n!$, the injection is a bijection.

Consequently, T_i^p maps the fiber F_λ bijectively onto $F_{T^p(\lambda)} = F_\lambda$. Thus T_i^p is a permutation of the finite set F_λ . Any permutation of a finite set has finite order, so there exists $m \geq 1$ such that $(T_i^p)^m(\pi^*) = \pi^*$. Therefore π^* is periodic under T_i . $\square$

The following is an immediate consequence of Proposition 3.4.

Corollary 3.5. *For $\pi^* \in \mathcal{P}_n^*$ and $i \in \{1, 2, 3, 4\}$, let k_i and k be the least nonnegative integers such that $T_i^{k_i}(\pi^*)$ and $T^k(S(\pi^*))$ are periodic, respectively. Then $k_i = k$. In particular, the maximum preperiod under each of the four actions is $D(n)$.*

3.2 Connections between T_1, T_2, T_3 and T_4

The four maps introduced above are related pairwise through a natural reversal operation, which we now define.

Definition 3.6. *For $\pi = (\pi_1, \dots, \pi_k) \in \mathcal{P}_n^*$, where $\pi_i = (\pi_{i1}, \dots, \pi_{in_i})$, define*

$$\text{rev} : \mathcal{P}_n^* \rightarrow \mathcal{P}_n^*, \quad \text{rev}(\pi) = (\text{rev}(\pi_1), \dots, \text{rev}(\pi_k)),$$

where $\text{rev}(\pi_i) = (\pi_{in_i}, \pi_{in_i-1}, \dots, \pi_{i1})$ for $1 \leq i \leq k$.

Proposition 3.7. *Let $\pi \in \mathcal{P}_n^*$. The following diagrams commute:*

$$\begin{array}{ccc} \pi & \xrightarrow{T_1} & T_1(\pi) \\ \text{rev} \downarrow & & \downarrow \text{rev} \\ \text{rev}(\pi) & \xrightarrow{T_4} & T_4(\text{rev}(\pi)) \end{array} \quad \begin{array}{ccc} \pi & \xrightarrow{T_2} & T_2(\pi) \\ \text{rev} \downarrow & & \downarrow \text{rev} \\ \text{rev}(\pi) & \xrightarrow{T_3} & T_3(\text{rev}(\pi)) \end{array}$$

i.e., $\text{rev} \circ T_1 = T_4 \circ \text{rev}$ and $\text{rev} \circ T_2 = T_3 \circ \text{rev}$.

Proof. Reversing a column after removing its bottom cell gives the same column as removing its top cell after reversal. Hence the columns remaining from π are identical in $\text{rev}(T_1(\pi))$ and $T_4(\text{rev}(\pi))$. The new columns also agree: both contain the original bottom cells in right-to-left order when read from bottom to top. Since reversal preserves column lengths and the left-to-right order of columns, the sorting and tie-breaking rules give the same arrangement in both constructions. Thus $\text{rev} \circ T_1 = T_4 \circ \text{rev}$.

The same argument applies to T_2 and T_3 , except that the original bottom cells appear in the new column in left-to-right order when read from bottom to top. Therefore $\text{rev} \circ T_2 = T_3 \circ \text{rev}$. $\square$

T_4 removes the top cell of each column in $\text{rev}(\pi)$, namely π_{i1} , and forms the new column $(\pi_{k1}, \dots, \pi_{11})$. Hence

Corollary 3.8. *For every periodic $\pi \in \mathcal{P}_n^*$, $|O_{T_1}(\pi)| = |O_{T_4}(\pi)|$ and $|O_{T_2}(\pi)| = |O_{T_3}(\pi)|$.*

Proof. By Proposition 3.7, rev sends the T_1 -orbit of π bijectively to the T_4 -orbit of $\text{rev}(\pi)$, and similarly sends the T_2 -orbit of π bijectively to the T_3 -orbit of $\text{rev}(\pi)$.

For each i , the map T_i commutes with every permutation of the labels, since its rules depend only on the positions of the cells. Consequently, any two labelings of the same shape have the same orbit size under T_i . Since π and $\text{rev}(\pi)$ have the same shape, the stated equalities follow. $\square$

Therefore, it suffices to study the orbit sizes of periodic $\pi \in \mathcal{P}_n^*$ under the actions T_1 and T_2 .

4 Initial analysis of the orbit sizes

In this section, we first determine the orbit size under T_1 , then focus on T_2 by developing a coloring algorithm on the Ferrers diagram. The coloring algorithm reveals the permutation σ_π induced by T_2^{m+1} on the labeled cells, and we prove that the colored regions correspond exactly to the cycles of σ_π . We then compute the order of σ_π explicitly in terms of the parameters m and m' , obtaining a bound on the orbit size under T_2 , with equality when the underlying shape has period $m+1$. The section concludes with several illustrative examples.

4.1 The orbit size of π under the action T_1

Theorem 4.1. *Let n and m be positive integers such that $\Delta_m < n \leq \Delta_{m+1}$. Then every periodic orbit of $\mathcal{P}_n^*$ under T_1 has size $\text{lcm}(1, 2, \dots, m+1)$.*

Proof. Suppose $\pi \in \mathcal{P}_n^*$ is a periodic partition with distinguishable cells. By Theorem 2.3, π contains cells in $\text{diag}[k]$ for $1 \leq k \leq m+1$. We track how these cells move under T_1 . The cell in position (i, j) moves to position $(i+1, j-1)$, for $j > 1$. The cell in position $(i, 1)$ moves to position $(1, i)$.

These moves cyclically permute the k positions of $\text{diag}[k]$, so each cell on this diagonal first returns to its original position after exactly k iterations. Since every diagonal $\text{diag}[k]$ with $1 \leq k \leq m+1$ contains at least one cell, and all labels are distinct, the orbit size is $\text{lcm}(1, 2, \dots, m+1)$. $\square$

4.2 The orbit size of π under the action T_2

To better describe the movement of individual cells under the action of T_2 , we introduce coordinates. For a partition π , let $\pi(i, j)$ denote the cell in the i^{th} column and j^{th} row of its Ferrers diagram. The following proposition describes the new location of this cell after applying T_2 .

Proposition 4.2. *Let n and m be positive integers such that $\Delta_m \leq n < \Delta_{m+1}$ and let $\pi \in \mathcal{P}_n^*$ be a periodic point. Then*

$$\pi(i, j) = \begin{cases} (T_2\pi)(i+1, j-1) & \text{if } j > 1, \\ (T_2\pi)(1, m+1-i) & \text{if } j = 1 \text{ and } \pi \text{ has } m \text{ cells in its bottom row,} \\ (T_2\pi)(1, m+2-i) & \text{if } j = 1 \text{ and } \pi \text{ has } m+1 \text{ cells in its bottom row.} \end{cases}$$

4.2.1 The permutation σ_π

Let $\pi \in \mathcal{P}_n^*$ be periodic. Since the period of $S(\pi)$ under T divides $m+1$, the configurations π and $T_2^{m+1}(\pi)$ have the same shape. Comparing the labels in corresponding positions therefore defines a permutation of $\mathcal{L}_\pi$.

Definition 4.3. *Let $\pi \in \mathcal{P}_n^*$ be a periodic partition with distinguishable cells, and let $m \in \mathbb{N}$ be such that $\Delta_m \leq n < \Delta_{m+1}$. Define $\sigma_\pi : \mathcal{L}_\pi \rightarrow \mathcal{L}_\pi$ by*

$$\sigma_\pi(\pi(i, j)) = (T_2^{m+1}\pi)(i, j).$$

Thus, $\sigma_\pi(a) = b$ means that after $m+1$ iterations of T_2 , the label b occupies the position initially occupied by a .

Example 4.4. *For $\pi = ((a, b, c), (d), (e))$ in Figure 7, we have $m = 2$ and $T_2^3(\pi) = ((c, d, b), (e), (a))$. Comparing the labels in corresponding positions gives $\sigma_\pi = (a c b d e)$.*

Recall that the order $\text{ord}(\sigma)$ of a permutation $\sigma \in S_n$ is the least positive integer k such that $\sigma^k = \text{id}$; equivalently, it is the least common multiple of the lengths of the cycles in its disjoint cycle decomposition.

The orbit of a partition $\pi \in \mathcal{P}_n^*$ under T_2 exhibits a dual periodicity: the Ferrers diagram returns to its original shape every $m+1$ iterations of T_2 , while the permutation σ_π governs the cyclic rearrangement of the labels within this fixed shape. We therefore obtain the following result.

Theorem 4.5. *Let $\pi \in \mathcal{P}_n^*$ be a periodic partition under T_2 , and let $m \in \mathbb{N}$ be such that $\Delta_m \leq n < \Delta_{m+1}$. Then*

$$|O_{T_2}(\pi)| \mid (m+1) \operatorname{ord}(\sigma_\pi).$$

Moreover, if the period of the underlying shape $S(\pi)$ under T is $\ell = m+1$, then

$$|O_{T_2}(\pi)| = (m+1) \operatorname{ord}(\sigma_\pi).$$

Proof. Because the rules of T_2 depend only on positions, successive applications of T_2^{m+1} act on the labels by successive powers of σ_π . Consequently, $T_2^{(m+1) \operatorname{ord}(\sigma_\pi)}(\pi) = \pi$, so the period of π divides $(m+1) \operatorname{ord}(\sigma_\pi)$.

If the underlying shape has period $m+1$, every return time of π is a multiple of $m+1$. Moreover, $T_2^{k(m+1)}(\pi) = \pi$ if and only if σ_π^k is the identity. The least such positive k is $\operatorname{ord}(\sigma_\pi)$, proving the equality. $\square$

For the configuration in Example 4.4, the underlying shape has period $m+1 = 3$ and $\operatorname{ord}(\sigma_\pi) = 5$. Theorem 4.5 therefore gives $|O_{T_2}(\pi)| = 3 \cdot 5 = 15$, in agreement with Figure 7.

4.2.2 Coloring algorithm

The goal of the following subsection will be to describe the order of σ_π . We begin by introducing notation that will be used throughout.

Definition 4.6. *Let $\pi \in \mathcal{P}_n^*$ be a periodic partition, and let $m \in \mathbb{N}$ be such that $\Delta_m \leq n < \Delta_{m+1}$. The cells lying on the $(m+1)$ -th diagonal, denoted $\operatorname{diag}[m+1]$, are called extra cells. Their number is*

$$m' = n - \frac{m(m+1)}{2},$$

where $0 \leq m' \leq m$.

A row (respectively, column) containing an extra cell is called a long row (respectively, long column), while a row (respectively, column) containing no extra cell is called a short row (respectively, short column).

We show that the cycle structure of σ_π is encoded by a coloring of the cells of π . In the following description of the coloring procedure, we assume $m' > 0$. When $m' = 0$, the diagonal $\operatorname{diag}[m+1]$ contains no extra cells, and every cell remains uncolored.

Let

$$E = ((i_1, h_1), \dots, (i_{m'}, h_{m'}))$$

be the ordered list of extra cells of π along the diagonal $\operatorname{diag}[m+1]$, with $i_1 < i_2 < \dots < i_{m'}$.

Define a map $X : \mathbb{N}^2 \times \mathbb{N}^2 \rightarrow \mathbb{N}^2$ by

$$X((a_1, b_1), (a_2, b_2)) = (a_1, b_2),$$

which returns the intersection of the column of (a_1, b_1) and the row of (a_2, b_2) .

For $0 \leq k \leq m' - 1$, define $X^k(E)$ inductively by $X^0(E) = E$ and

$$X^k(E) = X(X^{k-1}(E)),$$

where X is applied to consecutive pairs of entries in the input ordered list. That is, if

$$X^{k-1}(E) = ((c_1, d_1), \dots, (c_r, d_r)),$$

then

$$X^k(E) = (X((c_1, d_1), (c_2, d_2)), \dots, X((c_{r-1}, d_{r-1}), (c_r, d_r))).$$

Note that $X^{m'-1}(E) = ((i_1, h_{m'}))$.

We now define a coloring of the cells of π . Let $\{C_1, \dots, C_{\lceil m'/2 \rceil}\}$ be a set of distinct colors. For each $1 \leq r \leq \lceil m'/2 \rceil$, we perform an *initial coloring* by assigning color C_r to the cells in

$$X^{r-1}(E) \cup X^{m'-r}(E).$$

After all initial colors have been assigned, extend the color of each initially colored cell straight downward in its column and straight leftward in its row, stopping before the first previously colored cell or at the boundary of the diagram. Newly colored cells do not initiate further *propagation*.

We also use C_i to denote the color class consisting of all cells assigned color C_i by this procedure.

Example 4.7. Consider the colored partition in Figure 8 with $n = 31$, $m = 7$, and $m' = 3$. The diagonal $\text{diag}[8]$ contains the extra cells $(1, 8)$, $(4, 5)$, and $(7, 2)$.

We have

$$E = ((1, 8), (4, 5), (7, 2)), \quad X^1(E) = ((1, 5), (4, 2)), \quad X^2(E) = ((1, 2)).$$

Since $m' = 3$, we use $\lceil m'/2 \rceil = 2$ colors. The initial coloring is given by

$$C_1(\text{yellow}) : X^0(E) \cup X^2(E) = \{(1, 8), (4, 5), (7, 2), (1, 2)\}, \quad C_2(\text{blue}) : X^1(E) = \{(1, 5), (4, 2)\}.$$

The coloring is then extended downward in columns and leftward in rows, as shown in Figure 8.

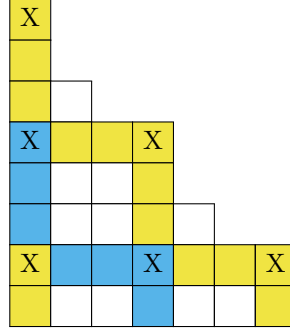


Figure 8: The cell coloring procedure applied to a $\pi \in \mathcal{P}_{31}^*$. The cells marked with ‘X’ indicate the initially colored intersection points before color propagation.

Example 4.8. For $n = 84$, $m = 12$, and $m' = 6$, consider the periodic partition whose extra cells are

$$E = ((3, 11), (4, 10), (5, 9), (6, 8), (9, 5), (12, 2)).$$

The initial coloring assigns C_1 (yellow) to $E \cup X^5(E)$, C_2 (blue) to $X^1(E) \cup X^4(E)$, and C_3 (orange) to $X^2(E) \cup X^3(E)$. The resulting coloring after propagation is shown in Figure 9.

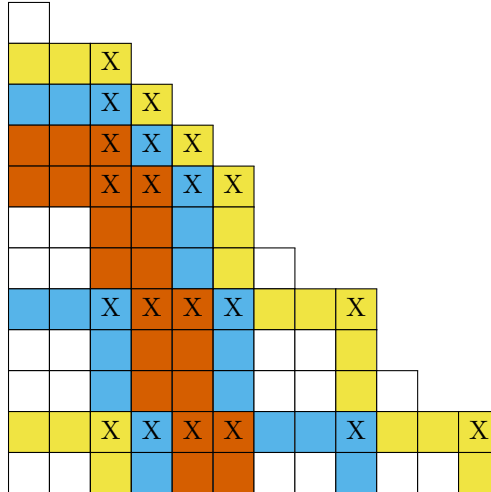


Figure 9: The cell coloring procedure applied to a $\pi \in \mathcal{P}_{84}^*$. The cells marked with ‘X’ indicate the initially colored intersection points before color propagation.

We finish this subsection with a couple of observations about the resulting coloring which will be useful in the forthcoming analyses.

First, some cells of π may remain uncolored by the coloring algorithm. In fact, a cell remains uncolored if and only if it lies in a short row and a short column of π . The total number of uncolored cells depends only on m and m' as given in the following lemma.

Lemma 4.9. *Let $\pi \in \mathcal{P}_n^*$ be a periodic point with $n = \Delta_m + m'$ where $0 \leq m' \leq m$. Then the number of uncolored cells in π under the coloring algorithm is exactly the triangular number*

$$\Delta_{m-m'} = \frac{(m-m')(m-m'+1)}{2}.$$

Proof. There are $m+1-m'$ unoccupied positions on $\text{diag}[m+1]$. Choose two such positions with column indices $a < b$. The cell $(a, m+2-b)$ lies in a short column and a short row, and it belongs to the Ferrers diagram because $a + (m+2-b) \leq m+1$.

Conversely, every uncolored cell (p, q) lies in a short column and a short row, so it determines two unoccupied positions on $\text{diag}[m+1]$ with column indices $a = p$ and $b = m+2-q$. Since (p, q) is not an extra cell, $p+q \leq m+1$, and hence $a < b$.

Thus uncolored cells are in bijection with pairs of unoccupied positions on $\text{diag}[m+1]$, and their number is $\binom{m+1-m'}{2} = \Delta_{m-m'}$. $\square$

Secondly, the initial coloring pairs the sets $X^t(E)$ and $X^{m'-1-t}(E)$ for $t = 0, \dots, \lceil m'/2 \rceil - 1$, from which the colors propagate. Each of these sets, together with the cells colored by propagation from it, forms a chain in the weak southeast order. When m' is even, each color class consists of two distinct chains (see Figure 9)). When m' is odd, each color class likewise consists of two distinct chains, except for the last color class, corresponding to $t = \lfloor m'/2 \rfloor$, for which the two chains coincide because $X^t(E) = X^{m'-1-t}(E)$. This occurs in Figure 8, where $m' = 3$ and the last color class consists of a single chain.

4.2.3 The permutation σ_π^{col}

We now relate the colored components of π to the cycle structure of σ_π induced by the action of T_2 . The coloring not only partitions the Ferrers diagram but also encodes the cycle decomposition of σ_π : each color class corresponds to one cycle, while the uncolored labels are fixed. To make this precise, we define a permutation σ_π^{col} determined by the coloring and show that $\sigma_\pi^{\text{col}} = \sigma_\pi$.

We define the permutation σ_π^{col} by specifying its disjoint cycle decomposition. For each color C_i , $1 \leq i \leq \lceil m'/2 \rceil$, we construct a cycle $\sigma_\pi^{C_i}$ as follows.

Using the heights $h_1, \dots, h_{m'}$ from E , set $l_i = h_{m'-i+1}$. To construct $\sigma_\pi^{C_i}$, first list the labels in the chain generated by $X^{i-1}(E)$ in weak southeast order, starting at $(1, h_i)$ and ending at $(m+2-l_i, 1)$. When the two chains are distinct, append the labels in the chain generated by $X^{m'-i}(E)$ in the same order, starting at $(1, l_i)$ and ending at $(m+2-h_i, 1)$. Read the resulting list cyclically, so that the final label maps to the first.

Thus, each $\sigma_\pi^{C_i}$ is a cycle beginning at $(1, h_i)$ and ending at $(m+2-h_i, 1)$.

Definition 4.10. *Let $\pi \in \mathcal{P}_n^*$ be a periodic partition with label set $\mathcal{L}_\pi$, and color its cells using the algorithm in Section 4.2.2. The colored permutation $\sigma_\pi^{\text{col}}: \mathcal{L}_\pi \rightarrow \mathcal{L}_\pi$ is defined by*

$$\sigma_\pi^{\text{col}} = \sigma_\pi^{C_1} \sigma_\pi^{C_2} \dots \sigma_\pi^{C_{\lceil m'/2 \rceil}},$$

where the cycles $\sigma_\pi^{C_i}$ are as constructed above. Every label of an uncolored cell is fixed.

Remark 4.11. *For each cell (p, q) , the following rules identify the label mapped to $\pi(p, q)$ under σ_π^{col} .*

1. **Fixed points:** *If row q and column p are both short, then*

$$\sigma_\pi^{\text{col}}(\pi(p, q)) = \pi(p, q).$$

2. **Downward $\downarrow$ move:** *If row q is short and column p is long, then*

$$\sigma_\pi^{\text{col}}(\pi(p, q+1)) = \pi(p, q).$$

3. **Diagonal $\searrow$ move:** If row q is long and column $p - 1$ is long ($p \geq 2$), then

$$\sigma_\pi^{\text{col}}(\pi(p - 1, q + 1)) = \pi(p, q).$$

4. **Rightward $\rightarrow$ move:** If row q is long and column $p - 1$ is short ($p \geq 2$), then

$$\sigma_\pi^{\text{col}}(\pi(p - 1, q)) = \pi(p, q).$$

5. **Jump $\circlearrowleft$ move:** If $p = 1$ and row q is long, then $q = l_i$ or $q = h_i$, and

$$\sigma_\pi^{\text{col}}(\pi(m + 2 - l_i, 1)) = \pi(1, l_i), \quad \sigma_\pi^{\text{col}}(\pi(m + 2 - h_i, 1)) = \pi(1, h_i).$$

Example 4.12. We illustrate the cycle construction using the colored partition in Figure 8, where $n = 31$, $m = 7$, and $m' = 3$. The coloring produces two classes: C_1 (yellow) and C_2 (blue).

For C_1 , the cycle begins at $(1, h_1) = (1, 8)$ and proceeds through the cells of color C_1 by successive southeast moves, passing through both colored components before returning to the starting point. This yields a cycle of length 14.

For C_2 , the cycle begins at $(1, h_2) = (1, 5)$ and similarly traverses all cells of color C_2 , forming a cycle of length 7.

Thus, σ_π^{col} decomposes into two disjoint cycles of lengths 14 and 7. The remaining (uncolored) cells are fixed points.

We now verify that $\sigma_\pi^{\text{col}} = \sigma_\pi$ in Example 4.12.

Example 4.13. Consider the colored partition $\pi \in \mathcal{P}_{31}^*$ with $m = 7$ and $m' = 3$, shown in Figure 10(a). The first $m + 1 = 8$ iterations of T_2 are displayed in Figure 10.

After 8 iterations (Figure 10(i)), the shape returns to its original form, and the permutation σ_π can be read off by comparing the labels with their initial positions. By Definition 4.3,

$$\sigma_\pi(\pi(i, j)) = (T_2^{m+1}\pi)(i, j).$$

From Figure 10(a,i), we obtain

$$\sigma_\pi = (1\ 2\ 3\ 6\ 7\ 8\ 12\ 16\ 22\ 23\ 24\ 31\ 18\ 25)\ (5\ 9\ 13\ 19\ 20\ 21\ 28).$$

This agrees with the coloring permutation obtained from tracing the colored components in Example 4.12, confirming that $\sigma_\pi = \sigma_\pi^{\text{col}}$ in this case.

Example 4.14. Consider the labeling in Figure 10(a) of the partition colored in Figure 8, where $n = 31$, $m = 7$, and $m' = 3$. The two yellow chains give a cycle of length 14, while the single blue chain gives a cycle of length 7:

$$\sigma_\pi^{\text{col}} = (1\ 2\ 3\ 6\ 7\ 8\ 12\ 16\ 22\ 23\ 24\ 31\ 18\ 25)\ (5\ 9\ 13\ 19\ 20\ 21\ 28).$$

All uncolored labels are fixed. Figure 10(b) shows $T_2^8(\pi)$. Comparing the labels in corresponding positions in the two panels gives the same permutation, verifying that $\sigma_\pi = \sigma_\pi^{\text{col}}$ in this example.

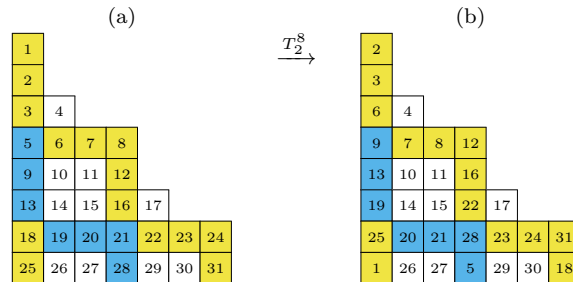


Figure 10: The configurations π in (a) and $T_2^8(\pi)$ in (b), with cells colored by the coloring algorithm. Comparing labels in corresponding positions gives the induced permutation σ_π .

4.2.4 $\sigma_\pi^{\text{col}} = \sigma_\pi$

We now prove that $\sigma_\pi^{\text{col}} = \sigma_\pi$. This requires analyzing the behavior of cells under the action T_2 and verifying that the coloring and cycle-tracing procedures capture the induced dynamics. We begin with a lemma describing the movement of individual cells.

Lemma 4.15. *Let $\pi \in \mathcal{P}_n^*$ be a periodic partition, and let a be a label in cell (p, q) of π . Under repeated applications of T_2 , the following hold:*

- (1) *If (p, q) is in a short row and a short column, then the label a remains in a short row and a short column.*
- (2) *If (p, q) is in a short row and a long column, then the label a remains in a short row and a long column until it reaches the bottom row, after which it moves to a long row and a short column.*
- (3) *If (p, q) is in a long row and column $p - 1$ is long ($p \geq 2$), then the label a remains in a long row until it reaches the bottom row, after which it moves to a long row and a long column.*
- (4) *If (p, q) is in a long row and column $p - 1$ is short ($p \geq 2$), then the label a remains in a long row until it reaches the bottom row, after which it moves to a short row and a long column.*

Proof. We analyze each case using Proposition 4.2 and the structure of periodic partitions.

Recall that a row (respectively, column) is short if it does not contain an extra cell, equivalently if there is an empty position on the diagonal $\text{diag}[m + 1]$ in that row (respectively, column).

(1) Suppose (p, q) lies in a short row and a short column. Then there are empty positions at $(m + 2 - q, q)$ and $(p, m + 2 - p)$.

For $q > 1$, the map T_2 moves the label a from (p, q) to $(p + 1, q - 1)$, while these empty positions move to $(m + 3 - q, q - 1)$ and $(p + 1, m + 1 - p)$, respectively. Thus a remains in a row and column that are both short.

If $q = 1$, the label a moves to $(1, m + 1 - p)$. The corresponding empty positions ensure that this row and column remain short, so the property is preserved.

(2) Suppose (p, q) lies in a short row and a long column. Then column p contains an extra cell at $(p, m + 2 - p)$.

For $q > 1$, the label a moves to $(p + 1, q - 1)$ and remains in a short row, and the extra cell moves to $(p + 1, m + 1 - p)$, so the column remains long.

If $q = 1$, the cell moves to $(1, m + 1 - p)$. Since the extra cell moves to $(p + 1, m + 1 - p)$, the row becomes long, while the first column is short. Thus the type changes to (long row, short column).

(3) Suppose (p, q) lies in a long row and column $p - 1$ is long. Then there is an extra cell at $(p - 1, m + 3 - p)$.

For $q > 1$, the image $(p + 1, q - 1)$ remains in a long row, since the extra cell structure propagates accordingly.

If $q = 1$, the cell moves to $(1, m + 2 - p)$. The extra cell at $(p - 1, m + 3 - p)$ moves to $(p, m + 2 - p)$, showing that this row is long, and the first column is also long. Hence the type becomes (long row, long column).

(4) Suppose (p, q) lies in a long row and column $p - 1$ is short. Then there is no extra cell at $(p - 1, m + 3 - p)$.

For $q > 1$, the image $(p + 1, q - 1)$ remains in a long row.

If $q = 1$, the cell moves to $(1, m + 2 - p)$. Since no extra cell moves into $(p, m + 2 - p)$, this row is short, while the first column is long. Hence the type becomes (short row, long column). $\square$

Now we are ready to state and prove the main result of this subsection.

Theorem 4.16. *Let $\pi \in \mathcal{P}_n^*$ be a periodic partition. Then*

$$\sigma_\pi = \sigma_\pi^{\text{col}}.$$

Proof. We fix a cell (p, q) of π and follow its label under $m + 1$ applications of T_2 . By Proposition 4.2, the label moves one step to the right and one step down until it reaches the bottom row, where it moves to the first column. We use Lemma 4.15 to determine which bottom-row rule applies and consider the five movement types in Remark 4.11.

The iteration counts in each of Tables 1–5 sum to $m + 1$. By Definition 4.3, the label originally occupying the final position maps to $\pi(p, q)$ under σ_π .

Case 1. Uncolored cells: row q and column p are both short.

Both bottom-row moves use the short-row rule. Table 1 gives the trajectory, which returns the label to (p, q) . Thus

$$\sigma_\pi(\pi(p, q)) = \sigma_\pi^{\text{col}}(\pi(p, q)) = \pi(p, q).$$

Initial position	Number of iterations of T_2	New position
(p, q)	$q - 1$	$(p + q - 1, 1)$
$(p + q - 1, 1)$	1	$(1, m + 2 - p - q)$
$(1, m + 2 - p - q)$	$m + 1 - p - q$	$(m + 2 - p - q, 1)$
$(m + 2 - p - q, 1)$	1	$(1, p + q - 1)$
$(1, p + q - 1)$	$p - 1$	(p, q)

Table 1: Position changes for Case 1.

Case 2. Downward move: row q is short and column p is long.

The first bottom-row move uses the short-row rule, while the second uses the long-row rule. As shown in Table 2, the final position is $(p, q + 1)$. Therefore,

$$\sigma_\pi(\pi(p, q + 1)) = \sigma_\pi^{\text{col}}(\pi(p, q + 1)) = \pi(p, q).$$

Initial position	Number of iterations of T_2	New position
(p, q)	$q - 1$	$(p + q - 1, 1)$
$(p + q - 1, 1)$	1	$(1, m + 2 - p - q)$
$(1, m + 2 - p - q)$	$m + 1 - p - q$	$(m + 2 - p - q, 1)$
$(m + 2 - p - q, 1)$	1	$(1, p + q)$
$(1, p + q)$	$p - 1$	$(p, q + 1)$

Table 2: Position changes for Case 2.

Case 3. Diagonal move: row q and column $p - 1$ are both long, with $p \geq 2$.

Both bottom-row moves use the long-row rule. Table 3 gives the final position $(p - 1, q + 1)$, so

$$\sigma_\pi(\pi(p - 1, q + 1)) = \sigma_\pi^{\text{col}}(\pi(p - 1, q + 1)) = \pi(p, q).$$

Initial position	Number of iterations of T_2	New position
(p, q)	$q - 1$	$(p + q - 1, 1)$
$(p + q - 1, 1)$	1	$(1, m + 3 - p - q)$
$(1, m + 3 - p - q)$	$m + 2 - p - q$	$(m + 3 - p - q, 1)$
$(m + 3 - p - q, 1)$	1	$(1, p + q - 1)$
$(1, p + q - 1)$	$p - 2$	$(p - 1, q + 1)$

Table 3: Position changes for Case 3.

Case 4. Rightward move: row q is long and column $p - 1$ is short, with $p \geq 2$.

The first bottom-row move uses the long-row rule, while the second uses the short-row rule. Table 4 therefore gives the final position $(p-1, q)$, yielding

$$\sigma_\pi(\pi(p-1, q)) = \sigma_\pi^{\text{col}}(\pi(p-1, q)) = \pi(p, q).$$

Initial position	Number of iterations of T_2	New position
(p, q)	$q-1$	$(p+q-1, 1)$
$(p+q-1, 1)$	1	$(1, m+3-p-q)$
$(1, m+3-p-q)$	$m+2-p-q$	$(m+3-p-q, 1)$
$(m+3-p-q, 1)$	1	$(1, p+q-2)$
$(1, p+q-2)$	$p-2$	$(p-1, q)$

Table 4: Position changes for Case 4.

Case 5. Jump move: $p=1$ and $q=l_i$ or $q=h_i$.

Since row q is long, the bottom-row move uses the long-row rule. Table 5 gives the final position $(m+2-q, 1)$. Hence

$$\sigma_\pi(\pi(m+2-q, 1)) = \sigma_\pi^{\text{col}}(\pi(m+2-q, 1)) = \pi(1, q).$$

Initial position	Number of iterations of T_2	New position
$(1, q)$	$q-1$	$(q, 1)$
$(q, 1)$	1	$(1, m+2-q)$
$(1, m+2-q)$	$m+1-q$	$(m+2-q, 1)$

Table 5: Position changes for Case 5.

These cases cover every cell of π . Thus the two permutations have the same preimage for every label, and hence $\sigma_\pi = \sigma_\pi^{\text{col}}$. $\square$

4.3 The order of σ_π^{col}

Having established that $\sigma_\pi^{\text{col}} = \sigma_\pi$, we now determine the order of this permutation. Since the order of a permutation is the least common multiple of its cycle lengths, this reduces to analyzing the cycle structure of σ_π^{col} . We do this by studying the sizes of the color classes obtained with the coloring algorithm.

Theorem 4.17. *Let $n \geq 1$, and let m be the unique positive integer such that*

$$\Delta_m \leq n < \Delta_{m+1}.$$

Set $m' = n - \Delta_m$, and let $\pi \in \mathcal{P}_n^*$ be periodic. Then

$$\text{ord}(\sigma_\pi) = \text{ord}(\sigma_\pi^{\text{col}}) = \begin{cases} 1, & \text{if } m' = 0, \\ m+1, & \text{if } m' = 1, \\ 2m-m'+3, & \text{if } 2 \leq m' \leq m. \end{cases}$$

Proof. Let $m' = n - \Delta_m$. We consider three cases.

Case 1: $m' = 0$. Then π is the full triangular partition of size Δ_m . In this case, σ_π^{col} , and hence also σ_π , is the identity permutation. Therefore,

$$\text{ord}(\sigma_\pi) = 1.$$

Case 2: $m' = 1$. There is exactly one extra cell, say at location (i, h) in the $(m+1)$ -st diagonal. The coloring algorithm colors this cell together with $i-1$ cells to its left and $h-1$ cells below. The total number of colored cells is

$$1 + (i-1) + (h-1) = i + h - 1 = m + 1.$$

These labels form a single cycle, so $\text{ord}(\sigma_\pi) = m + 1$.

Case 3: $2 \leq m' \leq m$. Let the extra cells on $\text{diag}[m + 1]$ be

$$E = ((i_1, h_1), \dots, (i_{m'}, h_{m'})), \quad i_1 < \dots < i_{m'},$$

so $i_k + h_k = m + 2$ and hence $h_1 > \dots > h_{m'}$. For $0 \leq t \leq m' - 1$ set

$$X^t(E) = ((i_1, h_{1+t}), \dots, (i_{m'-t}, h_{m'})).$$

Fix $1 \leq r \leq \lceil m'/2 \rceil$. The color C_r consists of the two components $X^{r-1}(E)$ and $X^{m'-r}(E)$ (coinciding if m' is odd and $r = \frac{m'+1}{2}$). Write

$$X^t(E) = ((c_1, d_1), \dots, (c_{m'-t}, d_{m'-t})).$$

Leftward propagation in row d_j colors $c_j - c_{j-1} - 1$ cells (with $c_0 = 0$), while downward propagation in column c_j colors $d_j - d_{j+1} - 1$ cells (with $d_{m'-t+1} = 0$). Thus, the total number of cells colored by C_r in this component, starting from the cells in $X^t(E)$, is

$$\sum_{j=1}^{m'-t} (1 + c_j - c_{j-1} - 1 + d_j - d_{j+1} - 1) = c_{m'-t} + d_1 - (m' - t) = i_{m'-t} + h_{1+t} - m' + t.$$

Applying this to the two components of C_r ,

$$\begin{aligned} |C_r| &= (i_{m'-r+1} + h_r - m' + r - 1) + (i_r + h_{m'-r+1} - r) \\ &= (i_{m'-r+1} + m + 2 - i_r - m' + r - 1) + (i_r + m + 2 - i_{m'-r+1} - r) \\ &= 2m - m' + 3. \end{aligned}$$

If m' is even, every color class has $2m - m' + 3$ cells. If m' is odd, the final color class consists of a single chain and has $(2m - m' + 3)/2$ cells, while every other color class has $2m - m' + 3$ cells. Since $m' \geq 2$, at least one cycle has length $2m - m' + 3$, and every other cycle length divides this number. Therefore,

$$\text{ord}(\sigma_\pi^{\text{col}}) = 2m - m' + 3.$$

The equality $\sigma_\pi = \sigma_\pi^{\text{col}}$ now gives the stated formula. $\square$

Corollary 4.18. *Let $\pi \in \mathcal{P}_n^*$ be a periodic point with $n = \Delta_m + m'$ and $0 \leq m' \leq m$. Then the orbit size of π under T_2 divides*

$$\begin{cases} m + 1, & \text{if } m' = 0, \\ (m + 1)^2, & \text{if } m' = 1, \\ (m + 1)(2m - m' + 3), & \text{if } 2 \leq m' \leq m. \end{cases}$$

Moreover, if the period of the underlying shape $S(\pi)$ under T is $\ell = m + 1$, then “divides” may be replaced by “equals”.

Proof. The claim follows from substituting the values of $\text{ord}(\sigma_\pi)$ from Theorem 4.17 into Theorem 4.5. $\square$

The following examples show that the bound may or may not be attained when the underlying shape has period less than $m + 1$.

Example 4.19 ($n = 18$, $m = 5$, $m' = 3$). *There are four periodic orbits in $\mathcal{P}_{18}$ under T , corresponding to necklaces of length 6 with 3 black beads. Figure 11 shows a labeled representative of each underlying orbit, together with its shape period ℓ .*

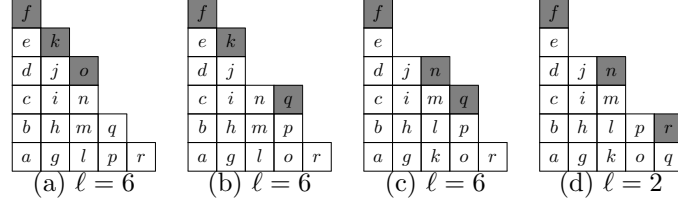


Figure 11: Labeled configurations whose underlying shapes represent the four distinct periodic orbits in $\mathcal{P}_{18}$ under T . Extra cells (the $m' = 3$ cells on $\text{diag}[6]$) are indicated in gray. The underlying shapes in (a)–(c) have period $\ell = 6$ under T , while the shape in (d) has period $\ell = 2$.

By Corollary 4.18, since $\ell = m + 1$, the first three orbits under the action of T_2 have size $(m + 1)(2m - m' + 3) = 60$, while the fourth orbit has size dividing 60. However, direct computation shows that the fourth orbit also has size 60.

Example 4.20 ($n = 8, m = 3, m' = 2$). There are two periodic orbits in $\mathcal{P}_8$ under T , corresponding to necklaces of length 4 with 2 black beads. Figure 12 shows a labeled representative of each underlying orbit, together with its shape period ℓ .

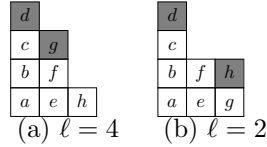


Figure 12: Labeled configurations whose underlying shapes represent the two distinct periodic orbits in $\mathcal{P}_8$ under T . Extra cells on $\text{diag}[4]$ are shaded gray and labeled d, g in (a) and d, h in (b). The underlying shapes in (a) and (b) have periods $\ell = 4$ and $\ell = 2$ under T , respectively.

Here $\text{ord}(\sigma_\pi) = 2m - m' + 3 = 7$. By Corollary 4.18, since $\ell = m + 1$ for orbit (a), its size under T_2 equals $(m + 1)\text{ord}(\sigma_\pi) = 28$; for orbit (b) with $\ell = 2$, the orbit size divides 28, and direct computation shows it is 14.

These examples illustrate the bound of Corollary 4.18: the orbit size under T_2 always divides $(m + 1)\text{ord}(\sigma_\pi)$, and sometimes attains it. In the next section, we determine the exact orbit sizes and prove that the upper bound $(m + 1)\text{ord}(\sigma_\pi)$ from Corollary 4.18 is achieved in most cases, even when $\ell \neq m + 1$, and when it is not, the orbit size is exactly half of that bound.

5 Exact sizes of orbits under T_2

In this section, we determine the exact orbit sizes of periodic partitions under T_2 and characterize when the bound $(m + 1)\text{ord}(\sigma_\pi)$ from Corollary 4.18 is attained. As the preceding examples show, equality can hold even when the period ℓ of the underlying shape is less than $m + 1$. We prove that, for $m \geq 3$, the bound is attained except when $m' = m - 1$ and $\ell = (m + 1)/2$, in which case the orbit size is exactly half of the bound. To establish these results, we analyze the permutation induced by one full period of the underlying shape, treating the colored and uncolored labels separately.

We begin by introducing notation for the colored and uncolored labels.

Definition 5.1. Let $\pi \in \mathcal{P}_n^*$ be a periodic partition whose cells are colored using the algorithm in Section 4.2.2. Define:

- $\mathcal{L}_{\pi,C}$ as the set of labels of colored cells of π ,
- $\mathcal{L}_{\pi,U}$ as the set of labels of uncolored cells of π ,

- for each color C_i ($i = 1, \dots, \lceil m'/2 \rceil$), $\mathcal{L}_{\pi, C_i}$ as the set of labels of cells colored with C_i .

Note that

$$\mathcal{L}_\pi = \mathcal{L}_{\pi, C} \sqcup \mathcal{L}_{\pi, U} \quad \text{and} \quad \mathcal{L}_{\pi, C} = \bigsqcup_{i=1}^{\lceil m'/2 \rceil} \mathcal{L}_{\pi, C_i}.$$

We now introduce the permutation of all labels induced by one full period of the underlying shape.

Definition 5.2. Let $\pi \in \mathcal{P}_n^*$ be a periodic partition. The permutation $\tau_\pi : \mathcal{L}_\pi \rightarrow \mathcal{L}_\pi$ is defined by

$$\tau_\pi(\pi(i, j)) = (T_2^\ell \pi)(i, j),$$

where ℓ is the period of the underlying unlabeled partition under T .

Lemma 5.3. Let $\pi \in \mathcal{P}_n^*$ be a periodic point with $n = \Delta_m + m'$, $0 \leq m' \leq m$. Let ℓ be the period of the underlying unlabeled partition under T and set $r = (m+1)/\ell$. For each color class C_i ($i = 1, \dots, \lceil m'/2 \rceil$) the following hold:

- (a) The set $\mathcal{L}_{\pi, C_i}$ is invariant under τ_π , i.e.,

$$\tau_\pi(\mathcal{L}_{\pi, C_i}) = \mathcal{L}_{\pi, C_i}.$$

- (b) The restriction $\tau_{\pi, C_i} := \tau_\pi|_{\mathcal{L}_{\pi, C_i}}$ is a permutation of $\mathcal{L}_{\pi, C_i}$ and satisfies

$$\tau_{\pi, C_i}^r = \sigma_{\pi, C_i},$$

where $\sigma_{\pi, C_i} := \sigma_\pi|_{\mathcal{L}_{\pi, C_i}}$.

- (c) The permutation τ_{π, C_i} is a single cycle on $\mathcal{L}_{\pi, C_i}$.

Proof. (a) We first show that a label retains its color after one application of T_2 , when the resulting configuration is colored by the algorithm.

Let (p, q) be a colored cell, and let

$$d = \#\{s : p \leq i_s \leq m+2-q\},$$

where $i_1, \dots, i_{m'}$ are the column indices of the extra cells. The color of (p, q) is $C_{\min(d, m'+1-d)}$. Indeed, at a cell of $X^{d-1}(E)$ the indicated interval contains exactly d extra-cell column indices, and propagation along the corresponding chain does not change this number.

If $q > 1$, Proposition 4.2 sends (p, q) to $(p+1, q-1)$. The interval and the extra-cell positions shift together, so d is unchanged.

Suppose $q = 1$. If the bottom row is long, the label moves to $(1, m+2-p)$, and the number of extra-cell column indices in the new interval is $m'+1-d$. If the bottom row is short, then column p must be long because the cell is colored. The label moves to $(1, m+1-p)$, and the new count is again $m'+1-d$. In both cases the resulting cell is colored, and

$$\min(d, m'+1-d) = \min(m'+1-d, d).$$

Thus the label retains its color.

Iterating this observation ℓ times, and using the fact that $T_2^\ell(\pi)$ has the same shape as π , shows that $\tau_\pi(\mathcal{L}_{\pi, C_i}) = \mathcal{L}_{\pi, C_i}$.

- (b) Therefore, the restriction

$$\tau_{\pi, C_i} := \tau_\pi|_{\mathcal{L}_{\pi, C_i}}$$

is a permutation of $\mathcal{L}_{\pi, C_i}$. Since $m+1 = \ell r$, By Definitions 4.3 and 5.2,

$$\tau_\pi^r = \sigma_\pi.$$

By Theorem 4.16, we also have $\sigma_\pi = \sigma_\pi^{\text{col}}$. Restricting this identity to $\mathcal{L}_{\pi, C_i}$ gives

$$\tau_{\pi, C_i}^r = \sigma_\pi|_{\mathcal{L}_{\pi, C_i}} = \sigma_\pi^{\text{col}}|_{\mathcal{L}_{\pi, C_i}} = \sigma_{\pi, C_i}.$$

- (c) By construction, each σ_{π, C_i} is a single cycle. Since $\sigma_{\pi, C_i} = \tau_{\pi, C_i}^r$, a power of τ_{π, C_i} is a single cycle. Hence τ_{π, C_i} itself must be a single cycle on $\mathcal{L}_{\pi, C_i}$.

This completes the proof. $\square$

To combine the contributions of the colored and uncolored labels to the orbit size, we will also need the following coprimality condition.

Lemma 5.4. *Let $\pi \in \mathcal{P}_n^*$ be a periodic partition with $n = \Delta_m + m'$ and $0 \leq m' \leq m$. Let ℓ denote the period of the underlying unlabeled partition under T , so that $\ell \mid (m+1)$, and set $r = (m+1)/\ell$. Then*

$$\gcd(\text{ord}(\sigma_\pi), r) = 1.$$

Proof. We consider three cases based on the value of m' .

Case $m' = 0$. Then, by Theorem 4.17, $\text{ord}(\sigma_\pi) = 1$, so $\gcd(\text{ord}(\sigma_\pi), r) = 1$.

Case $m' = 1$. Then, by Theorem 4.17, $\text{ord}(\sigma_\pi) = m+1$. Since there is a single extra cell, the underlying unlabeled partition has full period $\ell = m+1$, hence $r = 1$. Thus $\gcd(\text{ord}(\sigma_\pi), r) = 1$.

Case $2 \leq m' \leq m$. Then, by Theorem 4.17, we have $\text{ord}(\sigma_\pi) = 2m - m' + 3$. The binary word recording the occupied positions on $\text{diag}[m+1]$ consists of r repetitions of a block of length ℓ . Therefore its number m' of occupied positions is divisible by r . Write $m' = ra$ for some integer $a \geq 1$. Since $m = \ell r - 1$, we compute

$$\text{ord}(\sigma_\pi) = 2m - m' + 3 = 2(\ell r - 1) - ra + 3 = r(2\ell - a) + 1.$$

Hence $\text{ord}(\sigma_\pi) \equiv 1 \pmod{r}$, which implies $\gcd(\text{ord}(\sigma_\pi), r) = 1$. $\square$

We are now ready to state the main theorem that gives the exact orbit sizes under T_2 .

Theorem 5.5. *Let $\pi \in \mathcal{P}_n^*$ be a periodic point under the action T_2 , with $n \geq 1$ and $n = \Delta_m + m'$, $0 \leq m' \leq m$. Let ℓ be the period of the underlying unlabeled partition under T . Then, the orbit size of π under T_2 is*

$$|O_{T_2}(\pi)| = \begin{cases} (m+1) \cdot \text{ord}(\sigma_\pi), & \text{if } m - m' \geq 2, \\ \ell \cdot \text{ord}(\sigma_\pi), & \text{otherwise.} \end{cases}$$

Proof. Let $L = \text{ord}(\sigma_\pi)$ and $r = (m+1)/\ell$, and let τ_π be the permutation from Definition 5.2. Since $m+1 = \ell r$, Definitions 4.3 and 5.2 give $\tau_\pi^r = \sigma_\pi$.

By Lemma 5.3, τ_π preserves each colored label set $\mathcal{L}_{\pi, C_i}$. Consequently, it preserves their union $\mathcal{L}_{\pi, C}$ and its complement $\mathcal{L}_{\pi, U}$. Define the restrictions

$$\tau_c = \tau_\pi|_{\mathcal{L}_{\pi, C}}, \quad \tau_u = \tau_\pi|_{\mathcal{L}_{\pi, U}},$$

and let $m_c = \text{ord}(\tau_c)$ and $m_u = \text{ord}(\tau_u)$. Since these restrictions act on disjoint label sets,

$$\text{ord}(\tau_\pi) = \text{lcm}(m_c, m_u).$$

Moreover, σ_π fixes every uncolored label, so restricting $\tau_\pi^r = \sigma_\pi$ to $\mathcal{L}_{\pi, U}$ gives $\tau_u^r = \text{id}$. Hence $m_u \mid r$.

Every return time of π under T_2 is a multiple of ℓ , since its underlying shape has period ℓ . Because the shape repeats every ℓ steps and the rules of T_2 do not depend on the labels, $T_2^{k\ell}(\pi) = \pi$ if and only if $\tau_\pi^k = \text{id}$. Therefore,

$$|O_{T_2}(\pi)| = \ell \text{ord}(\tau_\pi) = \ell \text{lcm}(m_c, m_u).$$

By Lemma 5.3, each restriction τ_{π, C_i} is a single cycle on the same label set as σ_{π, C_i} , so these cycles have the same length. Since σ_π fixes all uncolored labels, taking the least common multiple of the colored cycle lengths gives $m_c = \text{ord}(\tau_c) = \text{ord}(\sigma_\pi) = L$. This also holds when $m' = 0$, since both permutations are identities.

Next, we determine $m_u = \text{ord}(\tau_u)$. Suppose first that there is no more than one uncolored cell. Then τ_u must be the identity on at most one label, hence $m_u = 1$.

Suppose now that there are at least two uncolored cells. Let $k = m - m'$. Let $A \subseteq \{1, \dots, m+1\}$ be the set of column indices of the unoccupied positions on $\text{diag}[m+1]$. Then $|A| = m+1 - m' = k+1 \geq 3$.

The uncolored labels are in bijection with the two-element subsets of A via

$$\pi(p, q) \longleftrightarrow \{p, m+2-q\}.$$

Indeed, an uncolored cell lies in a short column and a short row, and $p < m+2-q$. Conversely, for $a < b$ in A , the cell $(a, m+2-b)$ is uncolored.

By Proposition 4.2, if $q > 1$, the cell (p, q) moves to $(p+1, q-1)$, so its associated pair becomes $\{p+1, m+3-q\}$. If $q = 1$, the bottom row must be short because the cell is uncolored. Hence the second branch of Proposition 4.2 applies, sending $(p, 1)$ to $(1, m+1-p)$, whose associated pair is $\{1, p+1\}$. In both cases, the associated unordered pair is obtained by adding 1 modulo $m+1$ to each of its indices, with residues represented by $1, \dots, m+1$. Since the shape returns after ℓ steps, addition of ℓ modulo $m+1$ permutes A . Its induced action on the two-element subsets of A is the inverse of τ_u .

Suppose $\tau_u^d = \text{id}$. Then addition of ℓd modulo $m+1$ fixes every two-element subset of A . For any $a \in A$, choose distinct $b, c \in A \setminus \{a\}$. Since this translation fixes both $\{a, b\}$ and $\{a, c\}$, it fixes their intersection $\{a\}$. Thus $\ell d \equiv 0 \pmod{m+1}$, which implies $r \mid d$. Together with $\tau_u^r = \text{id}$, this proves that $m_u = \text{ord}(\tau_u) = r$.

We have shown that $m_c = L$, while $m_u = r$ if $m - m' \geq 2$ and $m_u = 1$ otherwise. By Lemma 5.4, $\gcd(L, r) = 1$. Consequently,

$$|O_{T_2}(\pi)| = \ell \text{lcm}(m_c, m_u) = \begin{cases} \ell L r = (m+1)L, & \text{if } m - m' \geq 2, \\ \ell L, & \text{otherwise.} \end{cases}$$

Since $L = \text{ord}(\sigma_\pi)$, this proves the theorem. $\square$

Combining Theorem 5.5 with the explicit formula for $\text{ord}(\sigma_\pi)$ from Theorem 4.17, we obtain the following closed-form expressions for orbit sizes when $m \geq 3$.

Theorem 5.6. *Let $\pi \in \mathcal{P}_n^*$ be a periodic point under the action T_2 , where $n = \Delta_m + m'$ with $m \geq 3$ and $0 \leq m' \leq m$. Let ℓ be the period of the underlying unlabeled partition under T . Then, the orbit size of π under T_2 is*

$$|O_{T_2}(\pi)| = \begin{cases} m+1, & \text{if } m' = 0 \ (n = \Delta_m), \\ (m+1)^2, & \text{if } m' = 1 \ (n = \Delta_m + 1), \\ (m+1)(2m - m' + 3), & \text{if } 2 \leq m' \leq m-2, \\ (m+1)(m+4) \text{ or } \frac{(m+1)(m+4)^*}{2}, & \text{if } m' = m-1 \ (n = \Delta_m + m-1), \\ (m+1)(m+3), & \text{if } m' = m \ (n = \Delta_m + m). \end{cases}$$

* The second value occurs exactly when $\ell = (m+1)/2$, equivalently when the two white beads in the associated necklace are opposite each other. This is possible only when m is odd.

Proof. For $0 \leq m' \leq m-2$, we have $m - m' \geq 2$, so the stated orbit sizes follow directly from Theorems 4.17 and 5.5.

Suppose $m' = m-1$. These theorems give $|O_{T_2}(\pi)| = \ell(m+4)$. The associated necklace has length $m+1$ and exactly two white beads. Any nonidentity rotation preserving its bead coloring must interchange the two white beads and is therefore a half-turn. Hence $\ell = m+1$ unless the white beads are opposite each other, in which case $\ell = (m+1)/2$. The latter is possible only when m is odd, giving the two stated orbit sizes.

Finally, if $m' = m$, the associated necklace has exactly one white bead and therefore has full period $\ell = m+1$. Since $\text{ord}(\sigma_\pi) = m+3$, Theorem 5.5 gives $|O_{T_2}(\pi)| = (m+1)(m+3)$. $\square$

6 Conjugation symmetries

In Section 3, we established how the reversal operation interacts with our four actions T_i , $i = 1, 2, 3, 4$: it conjugates T_1 to T_4 and T_2 to T_3 . Conjugation of Ferrers diagrams is another standard operation on partitions with distinguishable cells, so it is natural to ask how it interacts with these actions. Here we establish such relations for T_1 and T_2 on periodic partitions and use reversal to obtain corresponding relations for T_3 and T_4 .

Definition 6.1. Let $\pi \in \mathcal{P}_n^*$. The conjugate of π , denoted $\text{conj}(\pi)$, is the partition obtained by reflecting the Ferrers diagram of π across its main diagonal. Equivalently,

$$\text{conj}(\pi)(i, j) = \pi(j, i)$$

for every cell (i, j) in the Ferrers diagram of $\text{conj}(\pi)$. Thus, the label occupying row i and column j in π occupies row j and column i in $\text{conj}(\pi)$. Note that conjugation is an involution; that is, $\text{conj}(\text{conj}(\pi)) = \pi$ for every $\pi \in \mathcal{P}_n^*$.

Since conjugation preserves each diagonal $\text{diag}[k]$ setwise, Theorem 2.3 and Proposition 3.4 imply that it preserves periodicity. We examine how the actions T_1 and T_2 interact with conjugation.

Proposition 6.2. For every periodic $\pi \in \mathcal{P}_n^*$, the following diagrams commute:

$$\begin{array}{ccc} \pi & \xrightarrow{T_1} & T_1(\pi) \\ \text{conj} \downarrow & & \downarrow \text{conj} \\ \text{conj}(\pi) & \xleftarrow{T_1} & \text{conj}(T_1(\pi)) \end{array} \quad \begin{array}{ccc} \pi & \xrightarrow{T_2} & T_2(\pi) \\ \text{conj} \downarrow & & \downarrow \text{conj} \\ \text{conj}(\pi) & \xleftarrow{T_2} & \text{conj}(T_2(\pi)) \end{array}$$

Equivalently,

$$T_1 \circ \text{conj} \circ T_1 = \text{conj}, \quad T_2 \circ \text{conj} \circ T_2 = \text{conj}.$$

Proof. Let $\pi \in \mathcal{P}_n^*$ be periodic. Under conjugation, the label $x = \pi(i, j)$ moves to position (j, i) in $\text{conj}(\pi)$.

The action T_1 . Under T_1 , the label x moves to $(1, i)$ if $j = 1$, and to $(i+1, j-1)$ if $j > 1$. After conjugation, these positions become

$$(i, 1) \quad \text{if } j = 1, \quad \text{and} \quad (j-1, i+1) \quad \text{if } j > 1.$$

Applying T_1 once more, the entry at $(i, 1)$ moves to $(1, i)$, whereas the cell at $(j-1, i+1)$ moves to (j, i) . Thus in either case the final position is (j, i) , exactly the position of x in $\text{conj}(\pi)$. Therefore

$$T_1 \circ \text{conj} \circ T_1 = \text{conj}.$$

The action T_2 . Let k denote the number of columns of π , equivalently the length of its bottom row. Under T_2 , an entry with $j > 1$ again moves to $(i+1, j-1)$, while a bottom row entry at $(i, 1)$ is placed in the new first column at position $(1, k+1-i)$. After conjugation, these become

$$(k+1-i, 1) \quad \text{if } j = 1, \quad \text{and} \quad (j-1, i+1) \quad \text{if } j > 1.$$

Since π is periodic, the newly formed column of $T_2(\pi)$ has length k and is at least as long as every remaining column. Thus $T_2(\pi)$ has exactly k rows, so the bottom row of $\text{conj}(T_2(\pi))$ has length k . Hence applying T_2 again sends the entry at $(k+1-i, 1)$ to $(1, k+1-(k+1-i)) = (1, i)$, while the entry at $(j-1, i+1)$ moves to (j, i) . Thus the final position is again (j, i) , which is precisely the position of x in $\text{conj}(\pi)$. Therefore

$$T_2 \circ \text{conj} \circ T_2 = \text{conj}.$$

Since this holds for every entry x of π , the result follows. $\square$

On the set of periodic partitions, Proposition 6.2 can equivalently be written as

$$\text{conj} \circ T_i = T_i^{-1} \circ \text{conj}, \quad i = 1, 2.$$

Thus conjugation sends each periodic orbit to a periodic orbit of the same size and reverses the direction of the action.

Remark 6.3. *Combining Propositions 3.7 and 6.2 gives corresponding identities for T_3 and T_4 . On the set of periodic partitions,*

$$T_i \circ (\text{rev} \circ \text{conj} \circ \text{rev}) \circ T_i = \text{rev} \circ \text{conj} \circ \text{rev}, \quad i = 3, 4.$$

Indeed, these identities follow by composing the identities in Proposition 6.2 on both sides with rev and using Proposition 3.7. Thus, the involution $\text{rev} \circ \text{conj} \circ \text{rev}$ reverses the direction of the action on periodic orbits of T_3 and T_4 , just as conj does for T_1 and T_2 .

Remark 6.4. *Proposition 6.2 does not hold for arbitrary (non-periodic) partitions. For example, let $n = 3$ and consider the one-column partition $\pi = ((a, b, c)) \in \mathcal{P}_3^*$ (where the cells are listed from bottom to top). Then $\text{conj}(\pi) = ((a), (b), (c))$; in particular $\text{conj}(\pi)$ has three columns of size 1. On the other hand,*

$$T_2(\pi) = ((b, c)) \cup (a) = ((b, c), (a)), \quad \text{conj}(T_2(\pi)) = ((b, a), (c)),$$

and finally

$$T_2(\text{conj}(T_2(\pi))) = ((c, b), (a)) \neq \text{conj}(\pi).$$

Thus the identity for T_2 fails for this nonperiodic partition. Similarly,

$$T_1(\text{conj}(T_1(\pi))) = ((b, c), (a)) \neq \text{conj}(\pi).$$

Consequently, the periodicity assumption in Proposition 6.2 cannot be omitted.

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